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ABSTRACT

This book, the 10th volume in the Journal for Research in Mathematics Education (JRME) Monograph Series, discusses the geometry curriculum and investigates how elementary school students learn geometric concepts and how Logo programming and its turtle graphics might affect this learning. This volume also provides details on the development, structure, and content of the geometry curriculum, educational implications of the research, and the broad issues of Logo and geometry in mathematics education. (Contains approximately 155 references, 39 tables, and 34 figures.) (MM)

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Logo and Geometry

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Chapter 1

Logo and Geometry

Numerous assessments report a "failure of students to learn basic geometric concepts," especially geometric problem solving (Beaton et al., 1996; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Fey et al., 1984; Kouba et al., 1988; Stevenson, Lee, & Stigler, 1986; Stigler, Lee, & Stevenson, 1990). This poor performance is due, in part, to the current elementary and middle school geometry curriculum. There is little opportunity for the students to develop geometric problem solving, spatial thinking, or geometric concepts. It is no wonder that, after experiencing such an impoverished geometry curriculum in elementary and middle school, many high school students do not have the necessary geometric intuition and background for a formal deductive geometry course. Thus, a critical barrier to students' successful learning of geometry is the failure of standard elementary school geometry curricula to systematically facilitate students' progression from basic intuitions and simple concepts to higher levels of geometric thought.

In the Logo Geometry Project, we attempted to develop a research-based Grades K–6 geometry curriculum that addressed these deficits. Our project was one of six funded by the National Science Foundation under a special RFP, "Materials for Elementary School Mathematics Instruction," dealing with the use of technology. In this monograph, we discuss a series of studies that evaluate this curriculum and, further, investigate how elementary school students learn geometric concepts and how Logo programming in turtle graphics might affect this learning. We emphasize several interrelated studies that we conducted in the context of the Logo Geometry Project, along with relevant findings from our subsequent research provided for by two additional National Science Foundation (NSF) projects that built directly on these studies. The remainder of this chapter will describe problems in geometry learning, our theoretical and research background, and the organization of the monograph.

PROBLEMS IN GEOMETRY LEARNING

In its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) states that "spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world" (NCTM 1989, p. 48).[1] Unfortunately, according to extensive eval-

[1] These 1989 *Standards* were, of course, those on which the Logo Geometry Project was based. The more recent *Principles and Standards for School Mathematics*, or *Principles and Standards* (National Council of Teachers of Mathematics, 2000), on which Clements served as a member of the writing team, was not officially released at the time of this writing. It should be noted that the *Principles and Standards* emphasizes geometry even more at the early grades.

uations of mathematics learning, elementary and middle school students in the United States are failing to learn basic geometric concepts and geometric problem solving and are woefully underprepared for the study of more sophisticated geometric concepts and proof, especially when compared to students from other nations (Carpenter et al., 1980; Fey et al., 1984; Kouba et al., 1988; Stevenson et al., 1986; Stigler et al., 1990). For instance, fifth graders from Japan and Taiwan scored more than twice as high as students from the United States on a test of geometry (Stigler et al., 1990). Japanese students in both first and fifth grades also scored much higher (and Taiwanese students only slightly higher) than students from the United States on tests of visualization and paper folding. Data from the Second International Mathematics Study (SIMS) showed that in geometry, eighth and twelfth graders from the United States scored at or below the 25th international percentile (McKnight, Travers, Crosswhite, & Swafford, 1985; McKnight, Travers, & Dossey, 1985).

In the recent Third International Mathematics and Science Study (TIMSS) assessment, geometry and measurement were two of the three topics in which students from the United States performed significantly worse than the international average (National Center for Education Statistics, 1996). Geometry achievement did not improve much from grade to grade, presumably because this content area is not emphasized in elementary grades. Students from the United States performed poorly when applying geometry and measurement knowledge in problem-solving situations (Mullis et al., 1997).

As stated in the beginning of this chapter, a primary cause of this poor performance may be the curriculum in the United States. Standard elementary and middle school curricula emphasize recognizing and naming geometric shapes, writing the proper symbolism for simple geometric concepts, developing skill with measurement and construction tools such as a compass and protractor, and using formulas in geometric measurement (Porter, 1989; Thomas, 1982). These curricula consist of a hodgepodge of unrelated concepts with no systematic progression to higher levels of thought that are required for sophisticated concept development and substantive geometric problem solving. In addition, many teachers teach only a portion of the geometry curriculum that is available to them. Porter (1989), for instance, reported that in entire districts fourth-grade and fifth-grade teachers spent "virtually no time teaching geometry" (p. 11). When taught, geometry was the topic most frequently identified as being taught merely for "exposure"; that is, given only brief, cursory coverage. The SIMS data for the eighth grade indicate that teachers rated the "opportunity to learn" geometry much lower than any other mathematical topic (McKnight, Travers, Crosswhite et al., 1985). At the secondary level, the traditional emphasis has been on formal proof, despite the fact the students are unprepared to deal with it. As Usiskin (1987) noted:

> There is no geometry curriculum at the elementary school level. As a result, students enter high school not knowing enough geometry to succeed. There is a geometry curriculum at the secondary level, but only about half of the students encounter it, and only about a third of these students understand it (p. 29).

BACKGROUND

The aim of our project was to address deficiencies in current elementary school geometry instruction by developing a reconceptualized, research-based curriculum that takes advantage of the graphics-based computer programming language Logo. To explain this choice, we describe our theoretical foundations as we understood and applied them initially and then describe how these foundations suggest benefits of learning geometry with the Logo turtle. In Chapter 6, we discuss the implications of our research for these theories, including enhancements and revisions of the theories that we made both to maintain theoretical-empirical connections and to better understand students' understanding and learning of geometry.

Theoretical Perspectives

The two theoretical perspectives that informed our work were those of Piaget and of the van Hieles. The work of Piaget and Inhelder (1967) included two major themes related to geometry. First, the developmental organization of geometric ideas follows a definite order, and this order is more logical than historical. That is, topological relations (e.g., connectedness, enclosure, and continuity) are constructed first, followed by projective (e.g., rectilinearity) and Euclidean (e.g., angularity, parallelism, and distance) relations. At best, this theme has received mixed support (Clements & Battista, 1992b). A variety of geometric ideas appear to develop over time, becoming increasingly integrated and synthesized. Certainly, some Euclidean notions are present at an early age. Originally, these ideas are intuitions grounded in such actions as building, drawing, and perceiving. Children might develop actions that produce curvilinear shapes before they develop those actions that produce rectilinear shapes. Even young children have basic geometric intuitions that might be productively built on in the elementary school classroom.

The second Piagetian theme is that mental representations of space are constructed through the progressive organization of the student's motor and internalized actions. Thus, any such representation of space, or scheme[2], is not a perceptual "reading" of the spatial environment but is built up from prior active manipulation of that environment. This theme is supported by research (Clements & Battista, 1992b). Students' ideas about shapes do not come from passive looking. Instead, ideas about shapes evolve as students' bodies, eyes ... and minds ... interact with the environment. In addition, children need to explore shapes extensively to fully understand them—merely seeing and naming pictures of shapes is insufficient. Finally, children have to explore the components and attributes of shapes. The focus on attributes leads to the next theory.

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[2] In this paper, we define *scheme* as a mental network of relationships connecting concepts and processes in specific patterns. *Representation* is another term that should be clarified. Whenever this is not obvious from the context, we will explicitly differentiate between internal, mental representations, such as schemes and external representations, or inscriptions, such as drawings or written symbols.

The second theoretical perspective was that of Pierre and Dina van Hiele. According to their theory, students progress through levels of thought in geometry (van Hiele, 1986; van Hiele-Geldof, 1984). The van Hiele theory is based on several assumptions. First, learning is a discontinuous process characterized by qualitatively different levels of thinking. These levels progress from a Gestalt-like level the van Hieles call "visual" through increasingly sophisticated levels of description, analysis, abstraction, and proof. Second, these levels are sequential, invariant, and hierarchical. Progress through the levels is dependent on instruction, not age. Teachers can "reduce" subject matter to a lower level, leading to rote memorization. However, students cannot bypass levels and achieve understanding. Understanding requires working through certain instructional steps. Third, concepts implicitly understood at one level become explicitly understood at the next level. Fourth, each level has its own language; the uninformed teacher can easily misinterpret students' understanding of geometric ideas.

Students at Level 1 (visual) can only recognize shapes as whole images. A given figure is a rectangle, for example, because "it looks like a door." At this level, students do not think about the attributes, or properties, of shapes. Some researchers claim that there is an even earlier level at which children do not reliably distinguish circles, triangles, and squares from nonexemplars of those classes. Students at this level appear to be unable to form reliable mental images of these shapes (Clements & Battista, 1992b). Other researchers prefer to categorize these children merely as "beginning" Level 1.

Students at Level 2 (descriptive/analytic) recognize and characterize shapes by their properties. For instance, a student might think of a square as a figure that has four equal sides and four right angles. Properties are established experimentally by observing, measuring, drawing, and model making. Students find that some combinations of properties signal a class of figures and some do not; thus, the seeds of geometric implication are planted. At this level, students do not, however, see relationships between classes of figures. For instance, a student might believe that a figure is not a rectangle because it is a square. Many students do not reach Level 2 until middle school or even high school.

Students at Level 3 (abstract/relational) can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand and sometimes even provide logical arguments in the geometric domain. They can classify figures hierarchically (by ordering their properties), and they can give informal arguments to justify their classifications (e.g., a square is identified as a rhombus because it can be thought of as a "rhombus with some extra properties"). Students can discover properties of classes of figures by informal deduction. For example, they might deduce that in any quadrilateral the sum of the angles must be 360° because any quadrilateral can be decomposed into two triangles, each of whose angles sum to 180°. As students discover such properties, they feel a need to organize the properties. One property can signal other properties, so definitions can be seen not merely as descriptions but as ways of logically organizing properties. At this level, students can now understand, for example, that a square is a

rectangle. This logical organization of ideas is the first manifestation of true deduction. However, the students still do not grasp that logical deduction is the primary method for establishing geometric truths.

Students at Level 4 can establish theorems within an axiomatic system. They recognize the difference among undefined terms, definitions, axioms, and theorems. They are capable of constructing original proofs. That is, at this level they can produce a sequence of statements that logically justifies a conclusion as a consequence of the "givens."

The van Hiele theory also includes a progressive, five-step[3] model of teaching that moves students from one level of thinking to the next. In Step 1 (information), the teacher places ideas at the students' disposal. In Step 2 (guided orientation), students are actively engaged in exploring objects (e.g., folding, measuring) so that they encounter the principle connections of the network of conceptual relations that is to be formed. In Step 3 (explicitation), students are guided to become explicitly aware of their geometric conceptualizations so that they can describe these conceptualizations in their own language and are guided to learn traditional mathematical language. In Step 4 (free orientation), students solve problems whose solutions require the synthesis and utilization of those concepts and relations. The teacher's role includes the following: to select appropriate materials and geometric problems (with multiple solution paths); to give instructions to permit various performances and to encourage students to reflect and elaborate on these problems and their solutions; and to introduce terms, concepts, and relevant problem-solving processes as required. In Step 5 (integration), teachers encourage students to reflect on and consolidate their geometric knowledge, with an increased emphasis on the use of mathematical structures as a framework for consolidation so that eventually these consolidated ideas may be placed in the structural organization of formal mathematics. At the completion of Step 5, a new level of thought is attained for the topic.

In general, research supports the usefulness of the van Hiele levels for describing the development of students' geometric concepts (Burger & Shaughnessy, 1986; Clements & Battista, 1992b; Fuys, Geddes, & Tischler, 1988). The finding that most students from the United States are not progressing through the levels—but that such progression is possible given better curriculum and teaching—cannot be ignored. For example, most textbooks do not require students to develop higher levels of thinking as they progress through Grades K–12 (Fuys et al., 1988). Curriculum developers and teachers must enrich the geometry learning of our students by going beyond the traditional teaching of geometry in the United States. We took up this challenge with the Logo Geometry Project. We believed that the available theory and research were sufficiently developed to guide curriculum development in this domain (Clements, in press).

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[3] The van Hiele model is labeled both *stages* and *phases* at different points (van Hiele, 1986); they are called steps here to avoid confounding the various meanings of these terms. This confounding would have been most extreme in chapter 6.

How did we apply the two theoretical perspectives, of Piaget and the van Hieles, to this project? From the Piagetian perspective, we took the idea that students construct initial spatial notions not from passive viewing but from actions, both perceptual[4] and imagined, and from reflections on these actions (Piaget & Inhelder, 1967). The Logo turtle moves to create shapes, with students giving mathematical commands to direct the turtle's movements (Clements & Battista, 1992b). Furthermore, because these commands indicate "to the turtle" how to make components of shapes and how those components must be combined, Logo activities may facilitate students' progression to higher levels in the van Hiele hierarchy of geometric thinking. For example, with the concept of rectangle, students are initially only able to identify visually presented examples, a Level 1 (or visual[5]) activity (e.g., a shape is a rectangle if it "looks like a door"). Writing a sequence of Logo commands, or a procedure, to draw a rectangle "... allows, or obliges, the student to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible to reflection" (Papert, 1980a, p. 145). That is, students must analyze the spatial aspects of the rectangle and reflect on how they can build it from components. This leads to recognition of the figure's properties, an example of Level 2 (descriptive/analytic) thinking. Taking this to the next step, if asked to design a rectangle procedure that takes the length and width as inputs, students must construct a type of definition. Thus, they begin to build intuitive knowledge about the concept of defining a rectangle, knowledge that they can later integrate and formalize—a Level 3 (abstract/relational) activity. If challenged to draw a square or a parallelogram with their rectangle procedure, students may logically order these figures, another Level 3 activity. Further, Logo work aids in the generation of many examples of a concept. Overall, then, Logo may have the potential to encourage the construction of geometric properties characteristic of Level 2 thinking, something that textbooks typically do not do (Fuys et al., 1988).

Research on Logo and Mathematics Education

When the Logo Geometry Project began, empirical findings on Logo were ambiguous (Clements, 1985). Early reviews concluded that there were conflicting results about the effects of Logo on overall mathematics achievement. Experiments by Logo's developers generated positive reports (Papert, Watt, diSessa, & Weir, 1979). In the United Kingdom, low-achieving 11-year-old boys with 2 years of directed Logo programming experience improved to perform at the same level as

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[4] *Perceptual* is used here, consistent with Piaget's original formulation, as meaning phenomena or experiences that depend on sensory input, in contrast to those that are represented mentally (and thus can be "re-presented" imagistically without sensory support). Thus, perceptual should not be confused with the notion that we, with Piaget, reject—that of "immaculate perception" in which perceived objects are immediately registered in the brain.

[5] We use the term *visual* throughout this manuscript when it promotes consistency with the usual van Hiele interpretation, recognizing that it would be more accurate to use terms such as *spatial* (especially for nonsighted individuals) or *imagistic* in certain situations. Our suggested rephrasing for this level is presented in chapter 6.

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a control group on one general mathematics test but fell behind the control group on another test (Howe, O'Shea, & Plane, 1980). Other studies showed little positive effect on mathematics achievement (Akdag, 1985; Pea, 1983), although there were suggestive results for geometric concepts (Lehrer & Smith, 1986; Noss, 1987). Thus, when we began the Logo Geometry Project, it was unclear whether Logo could have significant positive effects on mathematics achievement. Therefore, our goal was to cast a wide net to examine the effects of a specially designed Logo environment on elementary school students' learning of geometry.

More recent reviews, conducted after our project was completed, generally have been positive, as illustrated by the following quotation from McCoy (1996):

> Logo programming, particularly turtle graphics at the elementary level, is clearly an effective medium for providing mathematics experiences ... when students are able to experiment with mathematics in varied representations, active involvement becomes the basis for their understanding. This is particularly true in geometry ... and the concept of variable" (p. 443).

In the area of geometry, research has focused on concepts of plane figures, especially students' levels of geometric thinking about those figures; angle and angle measurement; and motion geometry. We discuss each of these in turn.

Guided Logo experience can significantly enhance students' concepts of plane figures (Butler & Close, 1989; Clements, 1987). When asked to describe geometric shapes, students with Logo experience give more statements overall and more statements that explicitly mention geometric properties of shapes than students with no Logo experience (Clements & Battista, 1989, 1990; Lehrer & Smith, 1986). In one study, students were able to apply their knowledge of geometry better than a comparison group, but there was no difference in their knowledge of basic geometric facts. The researchers concluded that the use of Logo influenced the way in which students mentally represented their knowledge of geometric concepts (Lehrer, Randle, & Sancilio, 1989). In a similar vein, it has been shown that middle school students using Logo move to higher levels of conceptualizing and begin to integrate spatial and symbolic mental representations (Clements & Battista, 1988; Hoyles & Noss, 1988). Ninth-grade students may prefer learning with Logo. In a study conducted by Olive (1991), an end-of-semester discussion of their experiences in the Logo class revealed that students believed the class to be intriguing and exciting. These students reported being less frustrated and more involved in the Logo class than they were in their other classes. They believed that they had learned a lot of mathematics, especially about angles, quadrilaterals, and the Pythagorean theorem. Finally, they thought Logo had helped them with problem solving. However, for these students, success and sophistication in the Logo programming aspects of the tasks were necessary, but not sufficient, for success and sophistication with the geometric concepts involved in the tasks.

Similarly, Logo experience appears to affect students' ideas about angle significantly. Responses of control students in one study reflected common usage, such as describing an angle as "a line tilted," or no knowledge of angle. In comparison, the Logo students indicated more mathematically oriented conceptualizations, such

as "Like where a point is. Where two lines come together at a point" (Clements & Battista, 1989). Several researchers have reported that Logo experience has a positive effect on students' angle concepts (Clements & Battista, 1989; du Boulay, 1986; Frazier, 1987; Kieran, 1986a; Kieran & Hillel, 1990; Noss, 1987; Olive, Lankenau, & Scally, 1986). However, in some situations, benefits do not emerge until students have more than a year of Logo experience (Kelly, Kelly, & Miller, 1986–87).

Logo experiences may also foster some unintended conceptions of angle measure. For example, students may confuse angle measure with the amount of rotation along the path (e.g., the exterior angle in a polygon) or the degree of rotation from the vertical (Clements & Battista, 1989). In addition, concepts generated while working with Logo do not replace previously learned concepts of angle measure. For example, students' conceptions about angle measure and the difficulties they have coordinating the relationships between the turtle's rotation and the constructed angle have persisted for years, especially if not properly guided by their teachers (Clements, 1987; Cope & Simmons, 1991; Hoyles & Sutherland, 1986; Kieran, 1986a; Kieran, Hillel, & Erlwanger, 1986). In general, however, appropriately designed Logo experience appears to facilitate understanding of angle measure. After working with Logo, students' concepts of angle size are more likely to be mathematically correct, coherent, and abstract (Clements & Battista, 1989; Kieran, 1986b; Noss, 1987), while showing a progression from van Hiele Level 0 to Level 2 in the span of the Logo instruction (Clements & Battista, 1989). If Logo experiences emphasize the difference between the angle of rotation and the angle formed as the turtle traces a path, confusions regarding the measure of rotation and the measure of the angle may be avoided (Clements & Battista, 1989; Kieran, 1986b).

Research indicates that Logo experiences can also aid the learning of motion geometry and related ideas such as symmetry. As they were working with a Logo unit on motion geometry, students' movement away from van Hiele Level 0 was slow. There was, however, definite evidence of a beginning awareness of the properties of transformations (Olson, Kieren, & Ludwig, 1987). In another study, middle school students achieved a working understanding of transformations and used visual feedback to correct overgeneralizations when working in a Logo microworld (Edwards, 1991). Furthermore, Logo experiences may help develop notions of symmetry. Students as young as first grade have been observed using such mathematical notions as symmetry (Kull, 1986). In addition, symmetry concepts are learned by students involved in Logo through middle school (Edwards, 1991; Gallou-Dumiel, 1989; Olson, 1985). One student used a specially designed Logo symmetry microworld to learn such concepts and effectively transferred her mathematical understandings to a paper-and-pencil problem (Hoyles & Healy, 1997).

Generally, then, studies support the use of Logo as a medium for learning and for teaching mathematics.[6] Results especially support Logo as a medium for

[6] For more complete recent reviews, see Clements, 1999; Clements & Sarama, 1997; McCoy, 1996.

learning and teaching geometry (Barker, Merryman, & Bracken, 1988; Butler & Close, 1989; Clements & Meredith, 1993; Hoyles & Noss, 1987b; Kynigos, 1991; Miller, Kelly, & Kelly, 1988; Salem, 1989).

However, not all research has been positive. First, few studies report that students "master" the mathematical concepts that are the teachers' goals for instruction. Second, some studies show no significant differences between Logo groups and control groups (Johnson, 1986). Without teacher guidance, mere "exposure" to Logo often yields little learning (Clements & Meredith, 1993). Third, some studies have shown limited transfer. For example, the scores of students from two ninth-grade Logo classes did not differ significantly from those of control students on subsequent high school geometry grades and tests (Olive, 1991; Olive et al., 1986). One reason is that students do not always think mathematically, even if the Logo environment invites such thinking (Noss & Hoyles, 1992). For example, some students rely excessively on visual/spatial cues and avoid analytical work (Hillel & Kieran, 1988). This visual approach is not related to an ability to create visual images but to the role of the visual "data" (i.e., the students' perceptions) of a geometric figure in determining students' Logo constructions. Although helpful initially, this approach inhibits students from arriving at mathematical generalizations if overused. Further, there is little reason for students to abandon visual approaches unless teachers present tasks that can only be resolved using an analytical, generalized, mathematical approach. Finally, dialogue between teacher and students is essential for encouraging predicting, reflecting, and higher level reasoning.

In sum, studies showing the most positive effects involve carefully planned sequences of Logo activities. Appropriate teacher mediation of students' work with those activities is necessary for students to construct geometric concepts successfully. This mediation must help students forge links between Logo and other experiences and between Logo-based procedural knowledge and more traditional conceptual knowledge (Clements & Battista, 1989; Lehrer & Smith, 1986). Care must be taken that such links are not learned by rote (Hoyles & Noss, 1992). We attempted to create the *Logo Geometry* curriculum to provide just such a planned sequence of activities that incorporated teacher mediation. Our research goals were (a) to evaluate students' learning as they worked with this curriculum and, in that context, (b) to investigate how elementary school students learn geometric concepts and (c) to describe the role Logo can play in this learning. Because we worked closely with teachers who had to meet state and school curriculum frameworks, the *Logo Geometry* curriculum included and assessed a wide range of topics.

Given the constant state of change in the field of computers in education, an issue may arise about the perceived relevance of Logo-based curricula. Today, Logo plays a proportionately smaller role in schools than when these data were being collected, especially given the influx of other computer programs. Yet there are five reasons that the research reported here is significant. First, various versions of Logo are still being sold (and shareware versions downloaded) and used in classrooms. In

addition, numerous alternate forms of turtle geometry exist (e.g., "Maps and Movement," Education Development Center, 1993, is but one example of a genre of courseware). These programs, and other navigationally based instructional programs, require a research base. Second, students' work in Logo environments has much to teach us about learning in other computer and noncomputer environments that are not necessarily computational. Third, this project represents one vision of instituting the early *Standards* (NCTM, 1989) for geometry and thus complements evaluations of similar projects that primarily emphasized number (Hiebert, 1999). Fourth, the assessment items (e.g., the pretest items alone) provide a valuable description of elementary school students' geometric knowledge. Fifth, the research has implications for the development of mathematics education theory, particularly the van Hieles' theory of geometric thinking.

ORGANIZATION OF THE CHAPTERS

This chapter introduced the *Logo Geometry* research and development project and its rationale. Chapter 2 provides details about the development, structure, and content of the curriculum and describes the participants and methods of our research. Chapter 3 presents the results. Chapter 4 discusses these results and presents conclusions and educational implications of this research. This includes a focus on students' knowledge of geometry before instruction, providing a portrait of the geometric knowledge of 1,605 elementary students (Grades K–6) in two states.

Chapter 5 presents brief descriptions of software development and research that emerged from the Logo Geometry Project research corpus. That is, we used our experiences with the Logo Geometry Project to develop our own version of Logo, "Turtle Math," in another NSF curriculum development effort. The principles on which we based that development, and our research findings, are reviewed in chapter 5 by Julie Sarama, who led the research on Turtle Math.

Finally, chapter 6 presents implications of this work for the educational theory of geometry understanding and learning. That is, we reflect on the research reported in chapters 3, 4, and 5 to develop an extensive elaboration and revision of the van Hiele theory. Our goal is to ensure that the monograph goes beyond the Logo Geometry Project to discuss the broad issues of Logo and geometry in mathematics education.

Chapter 2

Logo Geometry Curriculum and Methodology

The Logo Geometry Project was conceptualized by the codirectors, on the basis of their own and others' research that indicated the potential of Logo turtle graphics for developing elementary school students' geometric competencies. In the past, this potential often remained unrealized because of a lack of curriculum structure.

Curriculum development was conducted over four years. During the first year, extant research and curricula in elementary school geometry were reviewed, a new geometry curriculum was developed, Logo activities were constructed to promote the objectives of that curriculum, and materials to guide teachers' use of the Logo curriculum were developed. This new curriculum was designed primarily to help students move from intuitive geometric notions to their formal counterparts, using Logo as a mediator. For example, Piaget stated that children "can only 'abstract' the idea of such a relation as equality on the basis of an action of equalization, the idea of a straight line from the action of following by hand or eye without changing direction, and the idea of an angle from two intersecting movements" (Piaget & Inhelder, 1967, p. 43). These actions form the intuitive beginning of the angle concept. When students intentionally walk, draw, and discuss angles, they make such intuitions explicit. When they program the Logo turtle to create angles, they must use symbolic code to express these notions and, more important, to quantify them. In this way, Logo mediates between the intuitive and the eventual formalization of the angle concept.

In the second year, the Logo-enhanced curriculum was field-tested by teachers who were involved in refining the materials. Then the materials were formatively evaluated and revised. In the third year, more extensive field tests and evaluation were conducted. The field tests of the second and third years constitute the main research data reported in this monograph. In the fourth year, the materials were revised and written in final form and disseminated by the Silver, Burdett & Ginn Publishing Company (Battista & Clements, 1991).

As we have noted, the project began with a comprehensive review and evaluation of existing elementary school geometry curricula and relevant research. From both of these, we created the theoretical foundation described in chapter 1. We also created a scope and sequence of geometric topics that were based on current school curricula. Then, we determined which topics in geometry curricula remain of central importance, which are overemphasized, which are not given adequate attention, and which are not currently represented but should be. Next, we analyzed

videotapes of children working in Logo environments—observing the students' successes and difficulties in dealing with geometric ideas—to determine which ideas might be effectively taught within a Logo environment.[7] In brief, appropriate Logo work can help elementary school and middle school students mentally construct the types of geometric properties that they need to learn, something that textbooks often fail to do (Fuys et al., 1988).

On the basis of this initial work, we constructed a new geometry curriculum sequence and associated Logo activities. This work will be described in detail later in this chapter; for now, we offer an overview. The curriculum is organized into three major strands: the Path strand, the Shapes strand, and the Motions strand. In the Path strand, the concept of path is taught explicitly, both off computer, through activities such as walking and drawing, and on computer, as a record of the turtle's movements. This concept of path is then used to organize beginning geometric notions. In the Shapes strand, students think about shapes—such as triangles, rectangles, and other polygons—as special types of paths. Angle and angle measure are included in this strand. The goal of the Motions strand is to use computer graphics to introduce students to geometric transformations and to help them construct cognitive "building blocks," such as mental rotation of shapes, that are important in dealing with spatial problems. Concepts of congruence and symmetry are explicitly addressed here as well.

To obtain feedback on our curriculum and its theoretical rationale, we presented our work to our advisory board, our fellow project directors, and colleagues at several professional meetings. Our advisory board consisted of six experts from the areas of mathematics, computer science, and mathematics education or psychology (Richard Brown, Kenneth Cummins, Robert Davis, Wallace Feurzeig, Alan Hoffer, and Grayson Wheatley), as well as two master teachers, Linda Hallenbeck and Fran Bickum. We revised and elaborated the initial outline on the basis of the advice of these groups.

We next obtained feedback from two preliminary pilot tests. In the first, two graduate assistants and one experienced teacher taught the geometric ideas to small groups of students from each of four grade levels—K, 2, 4, and 6. From videotapes of the lessons and teachers' daily journals, we determined the strengths and weaknesses of the materials and subsequently revised them.

The second pilot test was more elaborate. In this case, forty-two students from Grades K–6 attended 15 daily sessions of 80 minutes each in July of 1987. An experienced teacher taught the curriculum separately to students at three different grade ranges, K–1, 2–4, and 5–6. All students were pretested and posttested. In addition, case studies were conducted for two to four students at each grade range. As each pair worked on the curriculum materials, a graduate assistant assigned to that pair observed their work. Each graduate assistant's goal was to ascertain the

[7] For detailed discussions of deficiencies in the then-current curricula and how these deficiencies can be ameliorated using Logo, see Battista and Clements, 1986, 1987, 1988a; Clements and Battista, 1986, 1989.

students' geometric conceptualizations as the latter worked through the curriculum materials. Further, individual interviews of about 45 minutes duration were conducted with each of these case-study students. The purpose of the interviews was to probe the students' geometric ideas both before and after they experienced the curriculum. Finally, the codirectors observed the teacher's presentation of the lessons and the individual work of the case-study students. In daily debriefing sessions, the project staff discussed the graduate assistants' case studies, the teacher's evaluation of the lesson, and the codirectors' observations to evaluate the effectiveness of the curriculum activities.

From this second preliminary pilot test, information such as results of the pretesting and posttesting, interviews, case studies, and other observations was used to revise the materials again. The revision created a more complete curriculum including (a) background information; (b) lesson plans; (c) student activity sheets, homework sheets, and adjunct materials; and (d) Logo microworlds and utility procedures.

The next step in the development of the curriculum was to train teachers in the use of the curriculum and to involve them in refining the curriculum's objectives for each grade level. Previously, we had met with the administrators of five school systems (all work in the first two years took place near Kent State University [KSU] in Ohio). We asked these administrators to select 14 experienced teachers to study, revise, and test the curriculum. The selected teachers participated in a four-week course. During the first 2 weeks of this course, we introduced the teachers to the curriculum and its rational. During the last 2 weeks, the codirectors and teachers worked in grade-level groups, revising the materials to make them appropriate for the separate grades. Each group then produced a document that detailed the objectives and activities appropriate for their individual grade level. We synthesized the suggested curricular revisions and grade-level documents and produced grade-level specific versions of the curriculum.

During the following school year (1987–88), the 14 participating teachers implemented this updated version of the *Logo Geometry* curriculum, devoting approximately 15% of their instructional time for mathematics to it. The teachers evaluated the materials, both for their instructional soundness and for their effectiveness in teaching geometry, and suggested revisions. We visited each teacher's classroom to provide assistance, to assess the extent and manner of use of the Logo materials, and to talk with students about their work with the curriculum. In addition, we carried out an extensive assessment program consisting of paper-and-pencil tests and individual interviews with the students of the 14 participating teachers. Finally, the project team (ourselves and two graduate assistants) conducted monthly meetings to discuss the teachers' evaluations and revisions and to discuss how to better teach the curriculum. In the summer following that school year, the teachers met for 2 weeks to work in their grade-level groups. With our guidance, they synthesized their experiences and offered suggestions for revisions. Again, we used the resultant documents to make curricular revisions for the set of materials appropriate for each grade level.

In the third year, the same 14 teachers taught the curriculum for the second time. In addition, another group of 14 teachers from Buffalo, New York, who received only a week of training on the materials during the summer, taught the curriculum for the first time. Thus, the new set of teachers represented teachers not involved in the development of the materials. Finally, 28 teacher cohorts were identified by administrators, 14 at each site, as "control teachers" and, along with their students, participated in the assessment program. A complete description of the research methodologies for the field tests of the second and third years is provided in chapter 3. A final summer meeting brought the teachers at each site together to make final suggestions for revisions of the materials.

Finally, in the fourth year we began an analysis of all existing assessment data to complement the teachers' suggestions. We worked with our editors at Silver, Burdett & Ginn to make final revisions and publish the materials as *Logo Geometry*.

LOGO GEOMETRY: CURRICULUM GOALS

Logo Geometry (*LG*) has three major curriculum goals: (a) achieving higher levels of geometric thinking, (b) learning major geometric concepts and skills, and (c) developing power and positive beliefs in mathematical problem solving and reasoning. These goals provide a undergirding for our specific research questions; that is, they are threads that we weave through our presentation of the results. We discuss each of these goals briefly here as a background for the next section, which describes the curriculum structure and activities.

1. Achieving Higher Levels of Geometric Thinking

One goal of the curriculum is to help students move toward higher levels of geometric thinking in the van Hiele hierarchy, as described in chapter 1.

2. Learning Major Geometric Concepts and Skills

The concepts and skills of this second goal include, but are not limited to, those encountered in standard curricula (e.g., shape concepts, but also concepts and skills involving geometric motions and paths). In addition, the curriculum emphasizes the students' development of the related mental processes, such as spatial visualization and estimation, and the abilities prerequisite for formal mathematical thinking, such as conjecturing and convincing. We have also emphasized the thoughtful use of computation and arithmetic operations in problem-solving situations.

Consistent with NCTM's first *Standards* (1989), *LG* "focuses on the development of relationships and understandings; it goes far beyond the common and narrow emphasis on naming and identifying shapes, listing their characteristics, and memorizing terms and definitions" (p. 42). This leads us to the discussion of the next major goal.

3. Developing Power and Positive Beliefs in Mathematical Problem Solving and Reasoning

Mathematics involves forming abstractions in ways that enable one to solve problems. Thus, in *LG,* we attempted to present each task as a problem situation and to have students build mathematical ideas within the overall context of solving problems. We attempted to develop such general problem-solving heuristics as making tables and looking for patterns. Moreover, in our observations of the teachers and in the case studies, we realized how the tasks were true and significant problems for the students, even though the tasks do not appear to be problematic to most adults.

CURRICULUM STRUCTURE AND ACTIVITIES

The curriculum is divided into three strands: Paths, Shapes, and Motions. In this section, we describe the structure and activities of these three strands, including activities for students and teaching strategies.

Path Activities

Because children understand beginning spatial notions in terms of action, and the mathematical concept of *path* can be thought of as a record of movement, it seemed natural to emphasize this concept in our beginning study of geometry within a Logo environment. For example, having students visually scan the side of a building, run their hands along the edge of a rectangular table, or walk a straight path might help them abstract the concept of straightness, but the concept of straightness can be brought to a more explicit level of awareness with Logo activities. It is easily demonstrated in Logo that a straight path is one that has no turning. Thus, we took the position that the concept of path should be explicitly taught and used as an organizing idea for beginning geometric notions. Below we discuss five components of the Paths strand.

Paths. In the first component of the Paths strand, students learn what a path is and how to identify various kinds of paths. They walk, then describe, several different interesting paths that have been laid out on the floor with masking tape, some of which are straight, some closed, some open, some with bends, and at least one with an arc or other "curvy part."

Students learn to construct paths in Logo by giving a sequence of movement commands. For example, a gridlike map appears with a designated starting and ending point for the turtle. The student is to direct the Logo turtle from the start to the finish. Students at the Grade K–1 level used a special Logo microworld called "Singlekey" that allowed them to enter most commands as a single key press (F for FD 10; R for RT 30).

Procedures for Paths. In the second component of the Paths strand, students analyze the relationship between Logo commands and the components of a path that

they create. A command such as FD 50 draws a straight path 50 units long. A command such as RT 90 creates a change in direction along the path. Students learn to write procedures in Logo and, consequently, their knowledge of different kinds of paths is reinforced. They write procedures to move the turtle through the maps encountered in the first component of the Paths strand described previously. In addition, students construct procedures for paths that have some, but not all, of the lengths of their straight components labeled in diagrams. Finally, students write procedures to draw closed and nonclosed, straight and nonstraight, simple and nonsimple paths.

Turns. In the third component of the Paths strand, students learn to specify and estimate an amount of a turn. (We believe that students understand the concept of *turn* earlier than the concept of *angle.* Thus, we introduce students to turns first, then, later, we relate turns to angles.) In the context of "playing turtle," the students are introduced to whole turns (all the way around), half turns, and quarter turns as a method of specifying an amount of turn. They are then introduced to degrees as the unit of turn that the turtle uses. Finally, students play two computer games that provide experience estimating angles and distances in a Logo environment. In the first game, each player gets two chances to turn the turtle toward a target, then two chances to move the turtle into the bull's-eye. In the second computer game, students navigate the turtle around an obstacle and into a target with the fewest commands possible. Estimation strategies, such as the use of reference measures, are emphasized.

Next, students have experience estimating the amount of turn and learn that an angle is a special kind of path. They predict the path the Logo turtle draws when issued a forward command, then a turn command, then finally another forward command, which results in an angle being drawn. The students then use the computer to check their estimate. This activity is designed to help students begin developing an intuitive understanding of the relationship between turns and angles.

Path/Command Correspondence and Debugging. In this single lesson, the fourth component of the Paths strand, students learn about two important topics in geometric problem solving within the Logo environment. The first is the correspondence between a Logo command and the component of a path that it creates (e.g., a FD 50 command creates a straight path 50 units long; a RT 60 command turns the turtle 60°). The teacher runs a series of commands and asks students to draw the path that would be created if one of these commands was changed. The second topic addressed is the ability to correct errors in Logo procedures (i.e., "debug" procedures). When students debug their code, they (a) reflect on the correspondence between Logo commands and components of the path these commands produce, (b) correct procedures containing errors instead of starting over, (c) gain an understanding of the nature of their errors and the ways they might be corrected, and (d) learn that this problem-solving technique of correcting procedures can be applied in many different contexts. A framework is introduced that provides a general four-step strategy for correcting one's errors during problem solving. Students debug several procedures using this framework.

Turtle Deliveries. In the fifth and final component of the Paths strand, students apply some of their previously learned skills in a problem-solving environment, and learn about the process of "undoing" (i.e., finding the inverse for) a sequence of actions. They are asked to write procedures that move the turtle from one point on a scene (depicted on the computer screen) to another (one of three restaurants) and then return the turtle to the starting point along the same path. The students must find a pattern for returning to the starting point (undoing the original commands) so that they can bring the turtle home even if the destination is off the screen (which requires replacing the last command with its inverse; e.g., LT 90 for RT 90, etc.). The teacher helps students connect the idea of "undoing a sequence of turtle commands" to the related idea of "undoing a sequence of arithmetic operations" by, for example, relating it to the following problem: "I am thinking of a number. If I multiply it by 2, then add 5, I get 13. What is my number?"

Shapes: Special Paths

Once students firmly grasp the concept of path, we move on to having them think about special paths such as squares and triangles. The goal of this second strand of the *LG* curriculum is to have students view these shapes as paths and thus begin analyzing the shapes in terms of their component components and properties. (The curriculum does not neglect the "shapes as wholes" perspective. Students identify shapes in the environment and describe and classify regions, including faces of solids, on the attribute of shape.)

Squares. In the first component of the Shapes strand, students learn to identify squares, construct them in Logo, describe their properties, and use them as building blocks to make other shapes. Students first identify examples of squares in the environment and then plan and write several Logo procedures to draw squares of various sizes. The teacher helps the students relate the components of the squares drawn to the corresponding components of the Logo procedures. The properties of a square are discussed. Finally, students use a general square procedure with inputs (variables) to produce creative designs that consist of several congruent squares.

Rectangles. In the second component of the Shapes strand, as with squares, students learn to identify rectangles, construct them in Logo, and describe their properties. Students identify examples of rectangles in the environment and then plan and write several Logo procedures to draw various rectangles. The teacher helps the students relate the components of the rectangles drawn to the corresponding components of the Logo procedures. The properties of a rectangle are discussed. Finally, students define a rectangle procedure with inputs. The students are asked if they can make various figures, such as squares, trapezoids, or parallelograms, with their rectangle procedure. Additionally, they are challenged to create designs. The characteristics of squares and rectangles, and the relationship between the two (i.e., a square is a special kind of rectangle), are summarized.

Equilateral Triangles. In the third component of the Shapes strand, students learn to identify triangles and describe their properties. In addition, they construct equilateral triangles in Logo and use them as building blocks to make other shapes. Students identify triangles in the environment and discuss their properties. They then figure out how to command the turtle to make an equilateral triangle. By using the problem-solving strategies of "guess and check" and "make a table," students are guided to determine the correct amount of turn in an equilateral triangle. Again, students generate creative designs or pictures.

Regular Polygons. In the fourth component of the Shapes strand, students learn about polygons, regular polygons, and identifying polygons by their number of sides. The students discuss the properties of regular polygons after engaging in a "guess my rule" activity in which the teacher sorts shapes one at a time into two categories (regular and nonregular polygons). Students determine the relationship between the number of sides in a regular polygon and the amount of each turn (i.e., the measure of the exterior angle). This idea is investigated in the context of the "rule of 360"; that is, in order for the turtle to turn all the way around, it must turn a total of 360°. Even though the rule of 360 is a powerful and general mathematical property, it is accessible to students in the elementary school grades.

Classifying Angles. In the fifth component of the Shapes strand, students learn that an angle can be thought of as the union of two rays with a common endpoint, how angles are measured in mathematics, and how to identify angles in polygons. One can think of an angle as a path created by a forward move, a turn, and another forward move. Up to this point in the curriculum, only the measure of this turn *along* the path has been considered. This lesson emphasizes the measure of a different—but related—turn, the one that moves one side of the angle onto the other. Students use the procedure ANGLE that draws a ray, turns the turtle the amount that is input, and then draws another ray.

Two computer games aid the students in establishing the relationship between (a) the measure of an angle and the amount of turn between the sides of the angle and (b) the amount of turn at a turning point in a path and the amount of turn between the sides of the angle thus created. In the first game, the computer draws a ray and, while the eyes of player 2 are closed, prompts player 1 to enter a turn measure. Because player 2's eyes are closed, only player 1 knows the amount of turn. The turtle rotates the given amount and draws another ray, forming an angle. Player 2 guesses the angle measurement. The difference between the actual turn and the estimated turn is the score of player 2. In the second game, the computer draws a path and prompts player 1 to enter a turn. Player 2 must enter another turn that heads the turtle back toward its starting point (i.e., the supplement of player 1's turn). As part of this component of the Shapes strand, the teacher initiates a discussion of the usefulness of 90° and 180° as reference angles in estimating angle measure and, finally, initiates a discussion of angle classification (i.e., as acute, obtuse, and right).

Interior Angles of a Polygon. In this problem-solving activity, the sixth component of the Shapes strand, students use Logo and previously discussed ideas from

the Shapes strand to determine the relationship between the number of sides of a polygon and the sum of its vertex angles. Students use the problem-solving strategy "look for a pattern."

Parallel Lines and Parallelograms. In the seventh component of the Shapes strand, students learn about parallelism and parallelograms in both standard and Logo contexts. A concept attainment lesson by the teacher introduces parallelograms and their properties. Afterward, students figure out how to draw parallel lines in Logo and write procedures for drawing parallelograms. Students make a table and attempt to determine the relationship between adjacent turns in a parallelogram.

Classification of Quadrilaterals. In the eighth component of the Shapes strand, students learn about the properties of parallelograms and the hierarchical relationships between parallelograms and other quadrilaterals. Given a parallelogram Logo procedure with inputs, the students are asked if they can make various figures, such as squares, trapezoids, and parallelograms, using only this procedure.

Motions

The goal of this third and final strand is for students to develop concepts in motion (transformational) geometry and to help students construct cognitive "building blocks" that are important in dealing with spatial problems. Davis (1984), for example, describes the cognitive building blocks that are needed to determine the area of a rotated square on a geoboard. In addition to mental images of squares and triangles, he cites mental representations of the acts of rotating and translating triangles, of putting them together to make other shapes, and even of cutting apart squares to get triangles. Fundamental to this strand are the ideas that there are an infinite number of figures congruent to a given figure and that these figures may be related by a combination of geometric motions (i.e., isometries of the plane).

Introduction to Symmetry. In the first component of the Motions strand, the teacher introduces the concept of symmetry with a "guess my rule" activity (in which shapes are sorted into the categories of *symmetric* and *not symmetric*). The students then investigate symmetry with Miras, learning how to determine whether a shape has a line or lines of symmetry.

Mirror. In the second component of the Motions strand, the students learn to predict mirror images and to construct symmetric figures. They first predict mirror images visually, then they use a MIRROR procedure in Logo to check their predictions. The MIRROR procedure draws a dotted symmetry line at the location and heading of the turtle and creates a "mirror turtle" with that location and heading. For each subsequent comment the students enter, the regular turtle executes the command and the mirror turtle executes the "symmetric command"; that is, directions of left and right are reversed, but all other properties remain. In this way, a path and the mirror image of that path are created. The students then use the MIRROR procedure to construct symmetric figures. Also, students examine the Logo code produced by MIRROR to learn a *path* perspective for constructing mirror

images; that is, the mirror image of a figure defined by a sequence of Logo commands is defined by that same sequence with the RIGHTs and LEFTs interchanged. This allows students to relate their visual notion of symmetry with the previously developed knowledge of paths. Finally, students explore what symmetric pictures and designs they can create with the MIRROR procedure.

Introduction to Geometric Motions. The goal of this activity, the third component of the Motions strand, is for students to develop the concept of *congruence*, to learn about the geometric motions of slide, turn, and flip, and to learn the related concepts of flip lines and turn centers. Prior to using these concepts in Logo, the teacher explains that two figures are congruent if they have the same size and shape (i.e., if and only if one fits on top of the other exactly). Next, the slide, flip, and turn motions are demonstrated on an overhead projector, and the students are asked to label the slide, flip, and turn images on a corresponding activity sheet. Students use a transparency to show the sliding motion for slides and, for turns, the turning motion and the turn center. Students use Miras to locate the flip (mirror) lines for flips. Motions must be specified by their mathematical parameters. For instance, to specify a turn, a student has to indicate the turn center and the amount of turn.

Logo Motions. There are several purposes for this activity, the fourth component of the Motions strand. Students learn how to perform planar motions with Logo on the computer. (This is accomplished via a MOTIONS microworld.) They also begin to acquire the ability to visualize the results of these motions. In addition, because students have to instruct the computer to perform these motions, they learn a formal way of specifying slides, flips, and turns. Thus, the computer serves as a transitional device from physical movements to more abstract mathematical conceptualizations. As such, it helps the students reach a higher level of geometric thinking in the van Hiele hierarchy.

Rocket. In the fifth component of the Motions strand, the students' spatial visualization skill and ability to use motions in a problem-solving environment are developed. Using the computer, the students specify how to move a rocket onto its landing pad with a sequence of slides, flips, and turns. In the process of this activity, the teacher discusses with the students effective strategies for finding a sequence of geometric motions that achieves a given goal. For example, students can often "see" that the rocket has to be turned, say, 90° to the right, but they have difficulty locating the turn center. The teacher encourages the student to first turn the requisite amount about any reasonable point, then SLIDE the rocket onto the pad. (In this way, the students are dealing with composition of motions.) In addition, in this activity, students gain further experience with spatial problem solving, and the use of slides, flips, and turns in a movable environment, when they use the computer motions to move the components of a robot face into proper position.

Congruence and Motions. In the sixth component of the Motions strand, the students learn that another way to think about congruence is that two figures are

congruent if, and only if, there is a sequence of slides, flips, or turns that moves one onto the other. In doing so, they formalize the intuitive notion that two figures are the "same size and shape" if one "fits on top of" the other. In this activity, students first determine visually whether two figures are congruent. They then check this visual estimate on the computer by attempting to find a sequence of motions that moves one figure onto the other.

Students also learn that congruent figures have "corresponding components" that are also congruent. They use the motions commands to place one figure onto a congruent one, then they record the corresponding components. The activity ends by asking students to draw conclusions about corresponding components of congruent polygons. Thus, students learn that the corresponding components of congruent figures are congruent.

Symmetry and Motions. In the seventh component of the Motions strand, the students learn that flips can be used both to create symmetric figures and to test whether figures are symmetric. In the first part of the activity, students create symmetric figures using flips on the computer. For example, students flip a triangle about one of its sides so that the triangle, together with its image, forms a symmetric figure. They draw the symmetric figure created and name the shape of the outline of that figure. In the second part of the activity, students test several figures to see whether they are symmetric. Using the computer, they determine whether they can flip the figure onto itself.

Motions as Flips. In the eighth and final component of the Motions strand, the students learn how slides and turns can be expressed as a composition of flips. Every slide is equivalent to two consecutive flips about parallel lines and conversely. Every turn is equivalent to two consecutive flips about intersecting lines and conversely. The intersection of these flip lines is the turn center.

This concludes the description of the three strands of the *LG* curriculum: Paths, Shapes, and Motions. Most teachers separate the strands by one or more months across a school year. In the next section, we describe the research methodology we used to evaluate this curriculum.

RESEARCH METHODOLOGY

The research reported here emphasizes the second and third years of the project. In this section, we describe the teachers and students who helped us field-test *LG* and the assessment procedures that we used during those two years.

Participants

For the 1987–88 field test, the participants were 324 students in Grades K–7.[8] All of the students were from the classes of *LG* teachers at Site 1 near Kent, Ohio.

[8] *LG* is for Grades K–6, but one selected teacher was forced to move to Grade 7, where she elected to teach the Grade 6 strands of *LG*. This applied only to the 1987–88 year.

(There was no control group for this year's field test.) These teachers were selected for the project by school administrators, who based their selections on the teachers' expertise in teaching mathematics and their willingness to participate. For the 1988–89 field test, there were two additional groups of teachers. First, a new group of *LG* teachers was added from the Buffalo, New York, area. Second, teachers in both states were matched with control teachers from the same grade and school. These teachers were selected by their administrators because they possessed equivalent experience, expertise, and mathematics knowledge to that of the *LG* teachers. These control group teachers were provided a copy of the goals and objectives for *LG* and agreed to teach these objectives. For the 1988–89 year, a total of 1,300 students in Grades K–6 were involved in Ohio (Site 1) and New York (Site 2). (Refer to Table 1 for the number of student participants by site and grade.) Each teacher identified three to five students to be interviewed; thus, we interviewed approximately eight students at each grade level at each site. The teachers chose students whom they judged to be of average mathematics achievement, to have maintained good attendance (no history of extended absences), and to be sufficiently confident to be interviewed individually.

Assessment and Data Collection

We used multiple assessment techniques, including pretests and posttests that sampled overall curriculum objectives, unit tests, structured interview scripts, and classroom observations. Each of these types of assessment is briefly discussed in the remainder of this chapter.

Pre-Post Geometry Achievement Test. The pre-post (PP) test was intended as a general test of geometric achievement and level of thinking and thus did not reflect knowledge or formats that were unique to *LG*. In this way, a reasonable comparison could be made between the *LG* curriculum and traditional curricula. In addition, the strengths and weaknesses of both traditional instruction and *LG* instruction could be identified. For both years, the pre-post (PP) test was administered to all students twice: once at the beginning of the school year (October) before any geometry instruction had been conducted and, again, at the end of the school year (May) after all geometry instruction had been concluded. The PP test was administered in two parts on consecutive days at each grade level, with only minor modifications (e.g., one item was administered only to Grades K–3 students, although a few items were not appropriate for students in lower grades). The items that required a written response were administered individually by the research staff to children in Grades K–2 (one Grade 2 class in 1987–88 read the items by themselves); students' verbal responses were transcribed. Throughout this monograph, we use the convention of labeling items for the two parts of the paper-and-pencil assessment as follows: PP/1.1 indicates the pre-post test, part 1, item number 1 (where the item number was the number of the item on the test as presented to students). When an item has several components, another number is added, as in PP/1.4.1, which indicates pre-post test, part 1, item number 4, the first question

(congruence of two circles). To aid the reader, the items are shown when they are discussed in the results.

Unit Tests. The unit tests were intended to measure the specific geometric achievement goals of *LG*. Teachers administered these tests only to *LG* students and the items usually differed by grade level (although contiguous grade levels sometimes shared items). Each unit test was administered following completion of the corresponding *LG* unit: Paths, Shapes, and Motions. Because these are criterion-referenced assessments without control group, pre-post, or grade-level comparisons, only brief conclusions from all tests are presented.

Interview. *LG* students were individually interviewed three times; before beginning the curriculum materials, after the completion of the Shapes strand (called the *interim interview* in this report), and after the completion of the entire curriculum. Control students (1988–89 only) were interviewed at the end of the year only (the postinterview). During each interview, a member of the project staff presented the students with geometric problems and probed their geometry knowledge and processes. All interviews followed standard protocols. These will be described and illustrated as they are discussed in the results.

Case Studies. In selected classrooms, the work of a pair of interview students was studied intensely. The pair was videotaped and observed during almost every class session. Videotapes were transcribed and interpreted in relation to the goals of the *LG* curriculum.

Classroom Observations. Project staff visited classrooms on a regular basis. As they observed the lessons, they focused their observations on interview students. They took field notes that were transcribed after the session. If the interview students worked on the computer during a classroom observation, this work was videotaped for subsequent analysis.

These procedures were followed during the field-test years, the second and third years of the project. The next chapter presents the results of these assessments.

Chapter 3

Results

The Logo Geometry Project developed a research-based Grades K–6 geometry curriculum. This chapter presents the findings for both field-test years, the 1987–88 and 1988–89 school years. Results for the paper-and-pencil assessments (PP and unit tests) are presented in broad conceptual categories. Because the PP test was almost the same across grade levels and was the only paper-and-pencil test administered to the control group, the presentation will be organized around pretest and posttest items, with items and data for these items presented within the body of this section. (Geometric figures included in any items are reduced for the sake of space.) Results for the unit tests will be discussed as they are relevant. Given the large number and complexity of statistics, summary statements (marked by *summary*) are presented for most results as a convenience to the reader. Data from the interviews and classroom observations follow the paper-and-pencil data within each broad category.

TOTAL ACHIEVEMENT

We begin the presentation of the PP data with several summary scores. Tables mentioned throughout this chapter can be found in the "Tables" section that begins on page 155.

K–6 Total

Certain items were included on the PP tests at every grade level from kindergarten to Grade 6.[9] (The tests for Grades 4–6 included additional items, which are discussed in the next section.) The sum of these items created the "K–6 Total" score (total possible score, 87; reliability coefficient, $r = .85$). Table 2 presents the means and standard deviations for the K–6 Total score by experimental group, grade, and site. A four-way repeated measures ANOVA on the 1988–89 scores revealed main effects for Treatment ($F(1, 1055) = 24.87$, $p < .001$), Time ($F(1, 1055) = 486.16$, $p < .001$), Grade ($F(6, 1055) = 81.95$, $p < .001$), and Site, ($F(1, 1055) = 7.05$, $p < .01$), and significant interactions for Treatment × Time ($F(1, 1055) = 64.36$, $p < .001$), Time × Grade ($F(6, 1055) = 6.38$, $p < .001$), Time × Site ($F(1, 1055) = 16.26$, $p < .001$), Grade × Site ($F(6, 1055) = 10.75$, $p < .001$), Time × Grade × Site ($F(6, 1055) = 9.63$, $p < .001$), and Treatment × Time × Grade × Site ($F(6, 1055) = 5.21$, $p < .001$). All other interactions were not significant.

[9] Recall that one selected teacher moved to Grade 7; her students received the Grade 6 assessments.

Post hoc Newman-Keuls tests on the Treatment × Time interaction revealed that the *LG* posttest score ($M = 71.63$) was significantly higher than all other scores (control posttest, 66.79; control pretest, 60.66; and *LG* pretest, 59.80), and the control posttest score was higher than both pretest scores ($p < .01$ for each comparison).

Post hoc analyses of the Treatment × Time × Grade × Site interaction for 1988–89 are summarized in Table 3. There are two reasons to attribute this interaction to differences between individual teachers and classrooms rather than to influences of teacher experience (Site 1 teachers were teaching *LG* for the second time). First, there is no interpretable trend across the grade levels. Second, in four of the five grades at which the two *LG* sites differed significantly, a priori observations revealed characteristics of the teachers and their practices that would explain the differences better than the construct of "teacher experience." For example, we had observed that the teachers in certain classrooms (Grades K and 1 at Site 1; Grade 5 at Site 2) were, from their first lessons, comfortable with the *LG* approach. They stated that the *LG* teaching strategies were consistent with those they use in all topics. In other classrooms, we observed that the teachers (both at Grade 4, Site 2) were new to computers and the Logo language and thus had difficulty adapting to new teaching demands. These findings and observations, together with a nonsignificant Treatment × Time × Site interaction ($F(1, 1055) = 0.03$, $p = .87$) and the scores from the 1987–88 field test, suggest that no general conclusion should be drawn that students whose teachers had two years rather than one year of experience teaching *LG* curriculum performed substantially differently on this paper-and-pencil test.

In summary, both the *LG* and control groups made significant gains on the K–6 Total geometry achievement test. In addition, the *LG* group scored significantly higher than the control group.

4-6 Total

Several items were included on the paper-and-pencil tests only for Grades 4 to 6. Thus, the total possible score of the complete paper-and-pencil pretests and posttests for these grades was 125 points (this includes the 87 points of the previous K–6 total). Table 4 presents means and standard deviations for the 4-6 Total test score (reliability coefficient, $r = .88$). A repeated measures ANOVA revealed main effects for Treatment ($F(1, 527) = 22.24$, $p < .001$), Time ($F(1, 527) = 1197.51$, $p < .001$), and Grade ($F(2, 527) = 56.53$, $p < .001$), and significant interactions for Treatment × Time ($F(1, 527) = 86.02$, $p < .001$), Grade × Site ($F(2, 527) = 34.25$, $p < .001$), Time × Grade ($F(2, 527) = 6.10$, $p < .01$), Time × Grade × Site ($F(2, 527) = 18.84$, $p < .001$), and Treatment × Time × Grade × Site ($F(2, 527) = 11.96$, $p < .001$). Post hoc analyses on the Treatment × Time interaction revealed that the *LG* posttest score (102.13 across grades) was higher than all other scores (control posttest, 94.31; control pretest, 82.21; and *LG* pretest, 81.01), and the control posttest score was higher than both pretest scores ($p < .01$ in each case).

Post hoc analyses of the Treatment × Time × Grade × Site interaction revealed only minor differences between grade levels at the two sites (e.g., only the sixth-grade control group of Site 1 did not make significant gains; the Site 2 *LG* score was significantly higher than the Site 1 *LG* score for sixth grade only).

In summary, both *LG* and control groups made significant gains on the 4-6 Total geometry achievement test, but the *LG* group scored significantly higher than the control group. Thus, on summary scores at both grade-level ranges, students who worked with the *LG* curriculum outperformed students who received traditional instruction.

To elaborate effects revealed by these summary scores, single items or coherent groups of items were submitted to repeated-measures ANOVAs or MANOVAs. For brevity's sake, main effects and interactions not directly relevant to the research questions (e.g., Time × Grade or those involving the sites) will not be discussed; rather, the Treatment × Time interaction and any other higher order interactions (e.g., significant Treatment × Time × Grade interactions) will be emphasized. Pertinent unit test items will be discussed within each conceptual category.

SHAPES AND LEVELS OF GEOMETRIC THINKING

As described in chapter 2, helping students achieve higher levels in the van Hiele hierarchy and reason about two-dimensional shapes were major goals of the *LG* curriculum. We hypothesized that while programming the turtle to draw geometric shapes students had to analyze the spatial aspects of the shapes and reflect on how these spatial aspects were related to commanding the turtle to move. Given the importance of these hypotheses, we collected substantial data related to them, including several paper-and-pencil items, interview items, and classroom observations.

Pre-Post Test

Identification of Triangles, Rectangles, and Squares. Three items dealt with identification of triangles, rectangles, and squares (see Figure 1 and Figure 2). They were designed by Burger and Shaughnessy (1986) to assess geometric levels of thinking regarding these shapes. One point was given for each correct identification. (The reader should assume this scoring procedure unless otherwise noted.) Therefore, for the triangles, the maximum possible score was 14; for the rectangles, 15; and for the squares, 15.

Table 5 presents the means and standard deviations for the responses to these three items. To simultaneously test differences between the groups of students, a doubly multivariate repeated-measures MANOVA was performed. Analyses revealed several significant omnibus interactions. The first was a Treatment × Time interaction (Pillais trace, $F(3, 1028) = 17.94$, $p < .001$). Two univariate tests were significant, the test for the triangles item ($F(1, 1030) = 45.80$, $p < .001$) and the test for the rectangles item ($F(1, 1030) = 21.83$, $p < .001$). Post hoc tests for both items revealed similar results: The *LG* posttest scores (11.90 and 10.87 for trian-

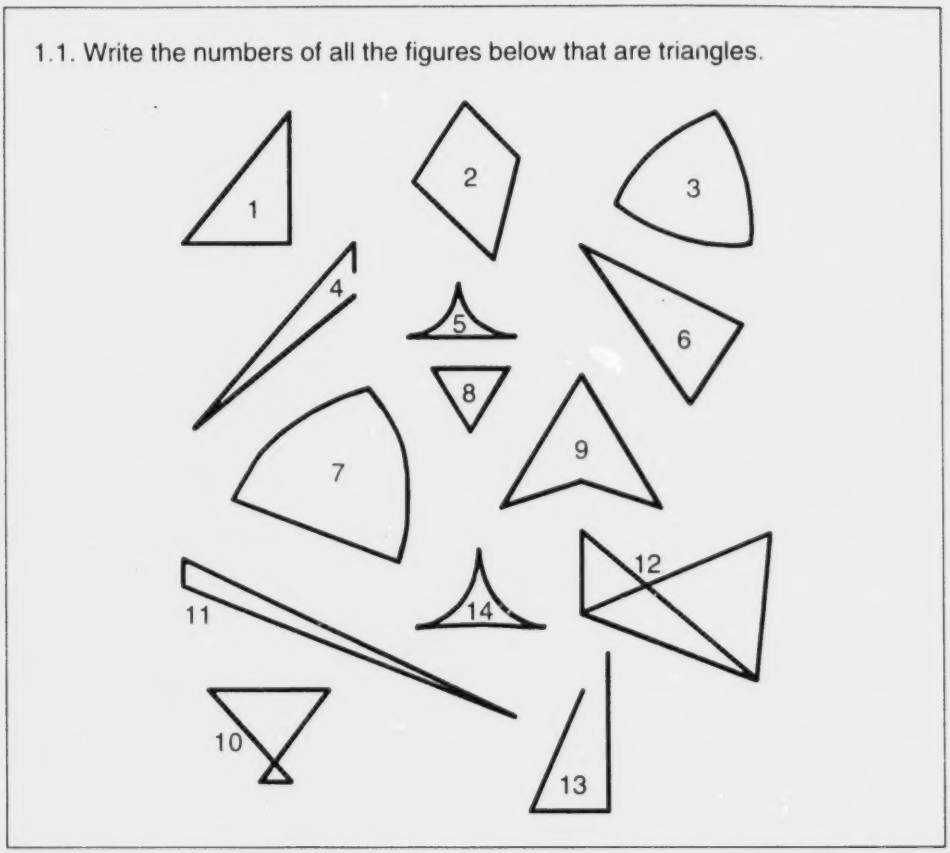


Figure 1.

gles and rectangles, respectively) were higher than all other scores (control posttest, 11.05, 10.09; control pretest, 10.29, 9.56; and *LG* pretest, 10.06, 9.46), and the control posttest scores were higher than both pretest scores ($p < .01$ for each comparison). Thus, for both items, both groups made significant gains in identifying triangles and rectangles; in addition, the *LG* group scored significantly higher than the control group.

For the second significant interaction, Treatment × Time × Grade (Pillais trace, $F(18, 3090) = 17.94$, $p < .001$), two univariate tests were significant, the test for the triangles item ($F(6, 1030) = 3.33$, $p < .01$) and the test for the squares item ($F(6, 1030) = 3.85$, $p < .01$). For the triangles item, the *LG* group performed better at the lower grades (e.g., K–2) and higher grades (5–6), when compared to the control group.

Results for the squares item were simpler. Post hoc tests revealed that the kindergarten *LG* group was the only group to make significant pre-post gains. (Means from the 1987–88 field test are consistent with this finding.)

Finally, for the Treatment × Time × Grade × Site interaction (Pillais trace, $F(18, 3090) = 3.02$, $p < .001$), only one univariate test was significant, the test for the

For the following items, you may use some numbers twice.
1.2. Write the numbers of all the figures below that are rectangles.
1.3. Write the numbers of all the figures below that are squares.

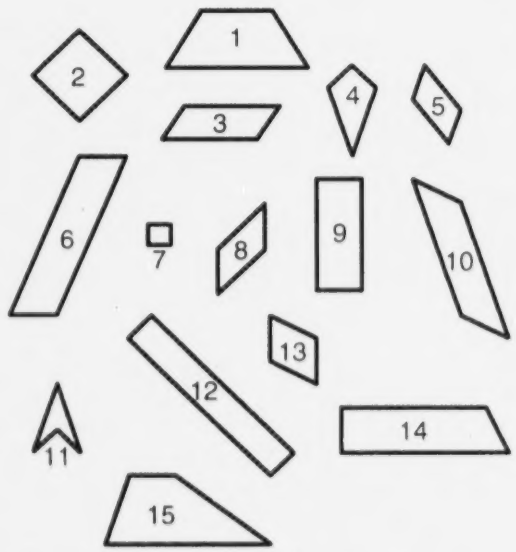


Figure 2.

triangles item ($F(6, 1030) = 6.91$, $p < .001$). Post hoc test results are summarized in Table 3. Consonant with the summary results, effects appear to be attributable to differences between teachers. That is, no interpretable pattern emerges that is not best explained by the strong teaching of individuals, such as the Grade 5 teachers at Site 2.

In summary, LG students performed better overall, with differences attributable to the triangles and rectangles items. For the triangles item, the *LG* group performed better relative to the control group at the lower and higher grades. Of all the groups, only the kindergarten *LG* group made significant gains on the square items.

Particular items, on which performance differed substantially between *LG* and control students, were investigated by examining means for individual figures within items. *LG* students, especially younger students, gained more than control students, not in their ability to identify figures that are triangles, but in their ability to correctly identify as nontriangles those figures that share spatial characteristics with triangles (e.g., the chevron, or deltoid). This was especially salient for the concave quadrilateral and for figures that appeared to be triangles but had curved sides or were not closed. There were few differences between the groups on triangles with large aspect ratios ("skinny") or prototypical (e.g., horizontal base, right; or equilateral) triangles.

For rectangles, large growth was displayed for *LG* students compared to control students for shapes 2 and 7, both squares. For both of these shapes, the most growth occurred in Grades 4, 5, and 6. Large relative gains were also shown for a right trapezoid (shape 14). Moderate growth was evinced for the nonrectangular parallelograms, especially shapes 5 and 8, in nonstandard orientation. Control instruction had a small negative effect for these shapes. Finally, small differences were observed for the remaining shapes, in some cases the result of a ceiling effect.

Attributing Properties to Figures. One item required students in Grades 2 to 6 to match statements about properties of figures, A–H, to two classes of figures, squares and rectangles (see Figure 3).

2.10. Write the letters of all statements that describe each type of figure. Some statements can be used to describe more than one type.

Statements:

A. Has 4 equal sides
B. Is a simple path
C. Is a closed path
D. Has opposite sides equal
E. Has 90-degree angles
F. Has 4 right angles
G. Has two long sides and two short sides
H. Requires 3 turns to trace if you start at the middle of a side

every square

every rectangle

Figure 3.

Table 6 presents means and standard deviations for this item. (The maximum possible score for 1988–1989 was 16; for 1987–1988, which included another question, "every triangle," it was 24.) A MANOVA revealed several significant interactions. First was a Treatment × Time interaction (Pillais trace, $F(2, 779) = 38.56$, $p < .001$). Both univariate effects, for "every square" and "every rectangle," were significant ($F(1, 780) = 70.33$, $p < .001$; $F(1, 780) = 23.75$, $p < .001$).

There was also a significant Treatment × Time × Grade interaction (Pillais trace, $F(8, 1560) = 3.04$, $p < .01$). Only one univariate effect was significant, that for rectangles ($F(4, 780) = 4.85$, $p < .01$). Means suggest that third-, fourth-, and fifth-grade students improved more in *LG* classrooms than in control classrooms. These differences were less salient for Grade 6 and for Grade 2; the control group made larger gains than the *LG* group.

In summary, LG students improved more than control students on attributing statements of geometric properties to the classes of square and rectangle. For the rectangle concept, effects tended to be strongest for students in Grades 3–5.

Examination of individual items indicated that *LG* students performed better than control students on properties dealing with paths (e.g., is a simple path; is a closed

path) and those dealing with angle measure (e.g., has 90-degree angles; has four right angles). For "opposite sides equal," *LG* students outscored control students for both shapes, the difference being small for rectangles but large for squares. The differences for the other properties were small and inconsistent.

Closed Figure: What Shape? The item illustrated in Figure 4 asked students in Grades 4 to 6 to consider what shape a closed figure might be (total possible score, 5).

2.11. I'm thinking of a closed figure with four straight sides.
It has two long sides and two short sides.
The two long sides are the same length.
The two short sides are the same length.

What shape could I be thinking of?

Circle Yes or No.

Could it be a triangle?	Yes	No
Could it be a square?	Yes	No
Could it be a rectangle?	Yes	No
Could it be a parallelogram?	Yes	No
Could it be a kite?	Yes	No

Figure 4.

In contrast to the results for Item 2.10, an ANOVA revealed that differences between the *LG* and control groups on the closed-figure Item 2.11 were insignificant (see Table 7).

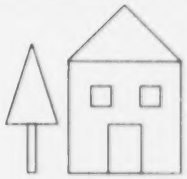
Disembedding. Disembedding of geometric figures was not included in the *LG* curriculum. Items that required disembedding were included because the assessments were designed for breadth and to reveal possible strengths of traditional instruction or detrimental effects of *LG*. Grades K–1 students were presented with a disembedding question in one of two forms, as illustrated in Figure 5. (The maximum possible score, 10 points, required students to identify squares as both "squares" and "rectangles.")

Table 8 presents the means and standard deviations for the total scores on the Grades K–1 disembedding questions. An ANOVA revealed a Treatment × Time interaction ($F(1, 256) = 8.73$, $p < .01$). On this item the *LG* group improved less than the control group.

Scores for the more complex disembedding item for students in Grades 2–6 revealed no significant differences on any main effect or interaction. *In summary*, there is no evidence that *LG* improved students' ability to disembed figures.

Length and Arithmetic: K–3 Building. Given the integration of geometry and number in *LG*, we expected that students would improve in applying arithmetic to geometric problems. Only primary-grade students were presented with the task illustrated in Figure 6.

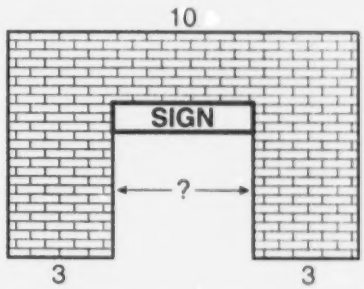
2.6 Kindergarten version:
Look at the picture.
Trace over all the triangles in blue.
Trace over all the rectangles in red.
Trace over all the squares in yellow.



2.6. First-grade version:
How many squares, rectangles, and triangles can you find in the picture?
triangles _____
rectangles _____
squares _____

Figure 5.

2.A. Here is a building. The builders want to build a sign.
How wide must the sign be to fit exactly?



Note: For Grades K–1, measures were as shown; for Grades 2–3, measures were multiplied by 10.

Figure 6.

Table 9 presents the means and standard deviations for this K–3 Building item, scored 1 or 0. An ANOVA revealed a Treatment × Time interaction ($F(1, 488) = 4.93$, $p < .05$). On this item, the *LG* group improved more than the control group.

There was also a significant Treatment × Time × Grade interaction ($F(3, 488) = 4.50$, $p < .01$). Kindergarten and third-grade students gained the most relative to their control groups. *In summary, LG* students scored significantly higher on items assessing students' recognition of the relevance of arithmetic processes in the solution of geometric problems and the accurate application of these processes, notably at kindergarten and Grade 3.

Unit Tests

Recall that unit tests were designed to measure the specific geometric achievement goals of *Logo Geometry*. Therefore, they were administered only to *LG* students, following completion of the corresponding *LG* unit. As such, they provide information on the degree to which goals were achieved. Unit tests included items on identifying shapes of various classes and on disembedding shapes.

Identification of Squares, Rectangles, and Triangles. About 75% of Grades K–1 students correctly identified squares, rectangles, and triangles when shown examples and nonexamples (e.g., three triangles, one kite, and one three-sided shape with a curved side). Second and third graders performed well when asked to identify these figures presented within a larger arrangement of the figures, though most did not identify squares as rectangles (accuracy ranging from 67% to 97%). Fourth graders performed similarly, with more identifying squares as rectangles.

Most first graders, given the names *squares, rectangles,* and *triangles,* identified squares and rectangles as having the same number of sides and stated that this number was four (69% to 95%). However, they were less able to give the number of turns in a square and a triangle without the benefit of a diagram (58% to 74%). Second graders were more accurate than these younger students in identifying the number of angles in figures for which diagrams were provided (69% to 98%, M = 86%); similarly, third and fourth graders accurately identified the number of turns and angles in a larger set of figures (50% to 96%, M = 80% and 50% to 95%, M = 87%, respectively).

Identification of Polygons and Regular Polygons. Intermediate-grade students were asked to identify various polygons with more than four sides. Fourth graders' performance was mediocre (44%) on a item in which they matched shapes to shape names, including such shapes as nonregular pentagons, hexagons, and octagons. These were not emphasized by the curriculum or teachers. In contrast, five sixth graders accurately (81%) identified all triangles, quadrilaterals, squares, rectangles, parallelograms, and hexagons from a set of figures.

Second through fourth graders were asked to identify regular polygons from a set of shapes. Second graders scored between 15% and 46% operating on a set of five figures. Third and fourth graders scored between 17% and 22% and 21% and 41%, respectively, on a set of 12 figures (for credit, students had to make no errors in identification across all figures). These same students gave an average of 1.32 and 1.55 (respectively) valid reasons why regular polygons are different from other polygons..

Disembedding. Kindergartners correctly identified triangles, squares, and rectangles embedded in pictures with about 85% accuracy. First graders correctly identified these shapes embedded in more complex figures with about 56% accuracy. Second graders, given only the most difficult of the two figures that were presented to first graders (the difficult item is shown in Figure 7), scored about 38% correct. *In summary,* this confirms that there is no evidence that *LG* improved students' ability to disembed figures.

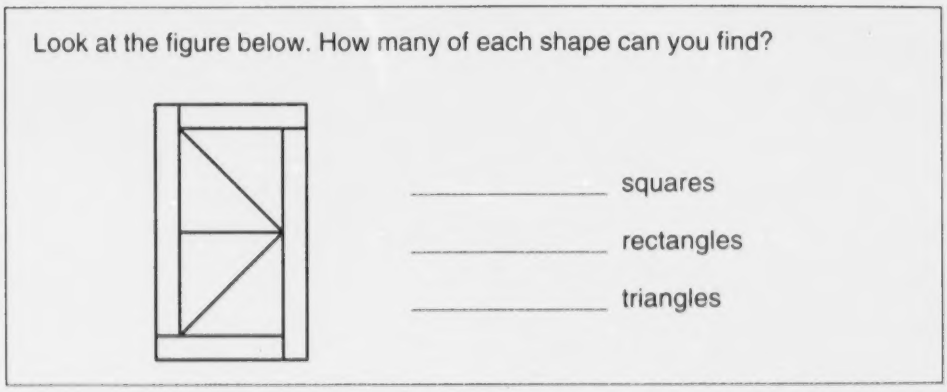


Figure 7.

Interview

Programming Rectangles. One interview question asked Grades 2–6 students, "How do you make a square using Logo?" (Figure 8). The number of students who responded correctly increased from 21% on the pretest to 85% on the posttest. Almost all students (95%) knew that all of the sides of a square were the same length. When asked on the posttest how they knew that all of the sides were the same length, 28% of the students justified their answer by referring to the Logo procedure (up from 13% on the pretest) and 38% by arguing that it wouldn't be a square if the sides were not all equal (up from 23%).[10]

On another interview task, students were asked, "How is shape 1 different from shape 2?"

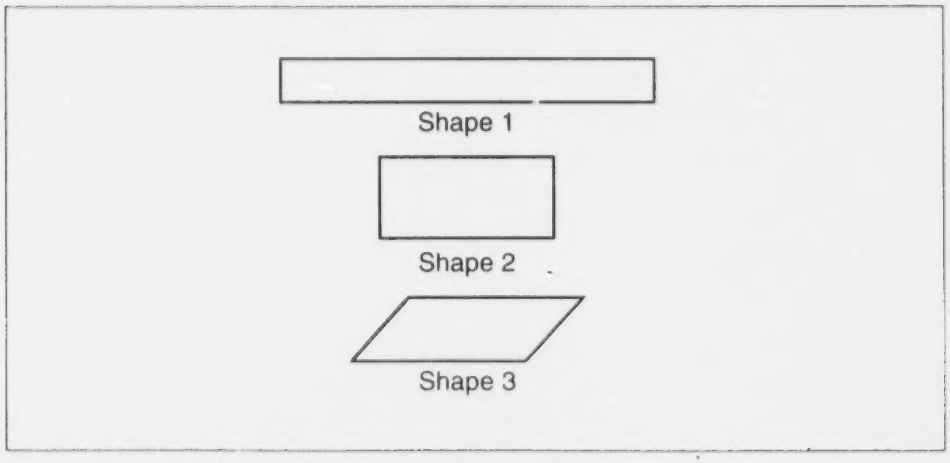


Figure 8.

[10] Because some questions were not asked of students who had not had some previous exposure to Logo, 47 students were asked this question on the pretest; 87 on the posttest.

The percentage of students stating that the side lengths were different increased from 3% on the pretest to 27% on the posttest. (Only Site 1 students were asked both times.) Students were then asked, "Here is a Logo procedure that draws shape 1. How would you change the procedure so that it would draw shape 2?" The number of students who gave a correct answer increased from 36% on the pretest to 73% on the posttest (students who did not know Logo were not asked the question). Students were then asked, "How would you change your procedure for shape 2 so that it would draw shape 3?" There was an increase in the percentage of students whose answers were basically correct (9%, 24%, 23% for pre/interim/post interviews). There was also an increase (20%, 29%, 35%) in the percentage of students who knew that the turns must be changed but did not estimate what those changes might be.

Triad Sorting. The interview included a triad polygon sorting task designed (with the help of Richard Lehrer) to determine the van Hiele level of geometric thinking for polygons. For each of nine items, children were individually presented with three polygons and asked, "Which two are most alike? Why?" For example, one student, presented with the three shapes labeled "2" in Figure 9, chose the middle and bottom figures, saying that they "looked the same, except that this one [the middle one] is bent in." She was attending to the visual aspects of the shapes; a Level 1 response. After working with *LG*, she chose top and middle figures, saying that they both had four sides. Thus, she let the overall visual aspect of the figures fade into the background, attending instead to the properties of each shape; a Level 2 response. Her conceptual structure had been reorganized not only to include, but to give prominence to, the corresponding *properties* of these figures. The illustrations in Figure 9 were used in the triad task. After we discuss the various scores on this task, we will discuss the various van Hiele levels indicated by the students' responses.

Triad Discrimination Scores. The reason students gave for choosing the pair in a triad as "more alike" was judged according to whether or not the reason discriminated that pair from the third item in the triad. Discrimination scores were not calculated if students gave a visual reason. However, if a student's reason was based on properties or a classification, the discrimination score was coded as follows:

0 What the student said was incorrect or did not discriminate the two shapes chosen from the one not chosen.

1 The student named more than one property, not all of which were correct, or discriminated the pair from the third figure.

2 The properties the student named were correct and correctly discriminated the two chosen shapes from the third shape.

A student's discrimination score was the sum of the discrimination scores received for individual triads. Table 10 shows the average triad discrimination scores of students at all grade levels for each of the three testing periods. Averages are for the groups shown. For example, 2.70 is the average discrimination score for Logo students (Grade K) on the postinterview.

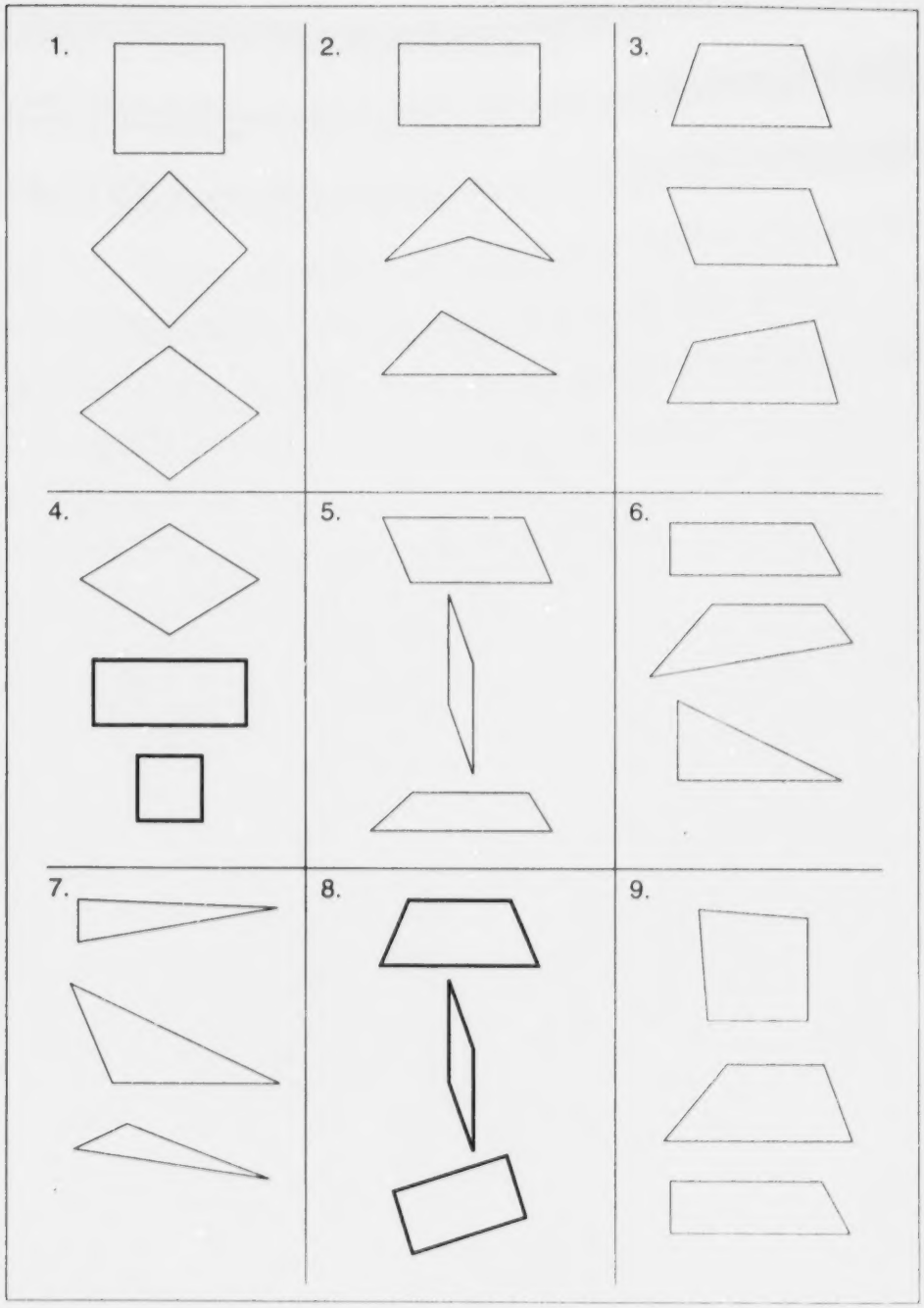


Figure 9.

Students in the higher grades had significantly higher discrimination scores than students in the lower grades, $F(6,197) = 20.47$, $p < .001$. Students' discrimi-

nation scores increased significantly from the pretest to the interim to the posttest $F(2, 140) = 13.49$, $p < .001$, and were higher, but not significantly higher, than those of the control group at the time of the posttest (all grades: Logo 2.66; control 2.05).

Triad Choices and Reasons. Students' reasons for choosing each triad pair were classified as visual (Level 1), property-based (Level 2), or classification (Level 3). See Table 11 for descriptions of the reasons and their levels. The percentage for each triad is summarized in Table 12. Overall, from the preinterview to the postinterview, there was a decrease in the average number of students at a grade level giving visual reasons for their choices (18%, from 86% to 68%) and an increase in the average giving properties and classification reasons (15%, from 10% to 25%). (The percentages do not add to 100 because some students occasionally gave no reason for their choices.) Although not shown in the tables, the average decrease in giving visual reasons for fifth and sixth grades was 18%, and the average increase in giving properties and classification reasons was 20%.

Triad van Hiele levels. Students' van Hiele levels were determined from the reasons that they gave for their triad choices. To be classified at a given level, a student had to give at least five (out of nine) responses at that level. If a student gave five responses at one level and at least three at a higher level, the student was considered to be in transition to the next higher level. If a student was classified as in transition from Level 1 to Level 2, the student was given a score of 1.5.

Table 13 shows the percentage of students at each grade level who were at the various van Hiele levels for (1) pretest; (2) interim, that is, after *LG* work with paths and shapes; and (3) posttest, that is, after the final unit of *LG* work that emphasizes geometric motions and related concepts. For 1988–89, the mean van Hiele levels for shapes for the *LG* group were pretest, 1.06; interim, 1.15; and posttest, 1.10. The mean van Hiele level for the control group on the posttest was 1.04. Although the mean posttest van Hiele level of the *LG* group was higher than that of the control group, it was not significantly higher. The mean van Hiele level of the *LG* group on the interim test was significantly higher than on the pretest ($t(112) = 2.67$, $p < .01$). Thus, we can conclude that, as measured by this task, the mean van Hiele level of the *LG* group was raised significantly by instruction on paths and shapes but that it declined by the posttest. (Students spent 2 weeks on paths and 2 weeks on shapes for Grades K–4 and 3 weeks total on paths and shapes for Grades 5 and 6.) For 1987–88, the mean van Hiele levels were pretest, .99; interim, 1.05; and posttest, 1.15, a significant increase $F(2,150) = 6.55$, $p = 002$. (Note: Two of the triad items were different for years 1 and 2.) As can be seen from Table 13, there was a shift to higher van Hiele levels from the pretest to the posttest interviews.

Overall, the percentage of students whose van Hiele levels increased from pretest to posttest was 32% for the Site 1 *LG* group in 1987–88 and 20% and 13% for the Site 1 and Site 2 *LG* groups, respectively, in 1988–89. Grade level increases can be found in Table 14.

Reasoning about Quadrilaterals. Students in Grades 4-6 were given the problem illustrated in Figure 10 to assess their ability to reason about geometric figures.

Five teams are involved in a geometry competition, and you are the judge for one of the problems. Each team is told that a mystery figure has four straight sides and is closed. The winner is the team that gives the smallest number of extra clues that will guarantee that the mystery figure is a rectangle. Here are the clues that each team gave. Which team's answer is best?

Team A	(1) two long sides and two short sides
Team B	(1) two long sides and two short sides (2) four right (90°) angles
Team C	(1) four right (90°) angles
Team D	(1) both pairs of opposite sides parallel (2) at least one right (90°) angle
Team E	(1) both pairs of opposite sides equal (2) at least one right (90°) angle

Figure 10.

As can be seen in Table 15, there was a dramatic decrease in the inappropriate choice of Team A (Site 1: 53%, 14%, 5%; Site 2: 33%, 12%, 13%). The increase for Team B (Site 1: 28%, 51%, 54%; Site 2: 29%, 60%, 58%) shows an increasing precision in students' expression of their geometric conceptualization of rectangle. (It must be noted that most students, on the basis of their past experience, do not consider a square to be a rectangle. So, this choice is probably an increase in precision over Team A.). The increase from pretest to interim for Team C, the correct choice, (Site 1: 8%, 20%; Site 2: 8%, 12%) is encouraging. However, the fact that students' performance decreased from interim to the posttest is consistent with the decrease in van Hiele levels during the same interval. Students seemed to have made progress in moving to more formal thinking about shapes during the Shapes module, but they regressed afterwards.

Students were then asked, "Which teams selected clues that would definitely make the mystery figure a rectangle?" For the Site 1 *LG* group, there was a substantial decrease in the number of students choosing Team A, the least precise of the descriptions (see the bottom of Table 15). There was a slight increase for Team B from pretest to posttest. (Note that, for many students on the pretest, this category contained redundant information.) There was a substantial increase for Team C on the interim, but it fell somewhat for the Site 2 group on the posttest. For both sites, there were noteworthy increases in the numbers of students who claimed that the clues of Team D and Team E were sufficient to guarantee a rectangle.

The students' postinterview reasons for not choosing a team's clues were also analyzed. Almost all of the students' reasons for not choosing Team A were valid (Site 1 *LG* 100%, Site 1 Control 96%, Site 2 *LG* 82%, Site 2 Control 77%). The control students were slightly more likely to draw a counterexample, whereas the *LG* students were slightly more likely to explain what was wrong with the clues.

Almost all students said that Team B's clues would make a rectangle, so there were not enough reasons for not choosing it to analyze. There was quite a difference between the reasons for not choosing Team C given by the *LG* group and the control group. The percentages of *LG* and control students who rejected Team C because "The clues don't say anything about the side lengths" were 70 and 15, respectively, for Site 1, and 17 and 19 for Site 2. The percentages of *LG* and control students who rejected Team C because "it could be a square or box" were 30 and 73, respectively for Site 1, and 83 and 63 for Site 2. So, the Site 1 *LG* students rejected Team C mostly because its clues did not mention side lengths, whereas the Site 1 control and Site 2 students rejected it because the clues did not eliminate squares.

There were few differences between the groups for rejecting Teams D and C, with most of the reasons referring to "it said only one angle had to be 90°." However, 16% of the students who rejected Team E did so because "it could be a square." Both of these reasons are consistent with the overall van Hiele level of the students. Students in Grades 4-6 were generally in transition from Level 1 (visual) to Level 2 (properties), so they were likely to abandon the visual condition for rectangles, namely, "two long sides and two short sides," and require that the figure have right angles. However, students still had difficulty with the square/rectangle classification problem and with the type of logical deduction required to deal effectively with the clues of Teams C, D, and E.

Classroom Observations

Classroom observations provided data that were useful in supporting and extending paper-and-pencil and interview data. One major hypothesis of the *LG* curriculum was that as path activities are extended to more complicated figures, students analyze the visual aspects of these figures, including how their components are put together, thus facilitating these students' transition from the visual to the descriptive-analytic level of geometric thinking. Observations also illuminated the mechanisms of this transition as the following episode demonstrates.

> A class of first graders was investigating the concept of rectangle. The students had identified rectangles in the classroom and had built them out of various materials, such as blocks, tape, clay, and geoboards. Then they went to the computer lab where they were asked to make the turtle draw rectangles. (These students were in the Singlekey environment, in which the turtle is given commands by typing single keys—F for forward 10 turtle steps, B for backward, R or L for right and left turns of 30°.)
>
> As the activity proceeded, all children were drawing rectangles in Logo. After drawing several rectangles with vertical sides, one student wanted to be different; he attempted to draw a rectangle that was tilted. He started by issuing an "R" command to turn the turtle. He then instructed the turtle to draw the first side using five Fs. He paused for quite some time as he came to the first turn, so one of the researchers asked him how much he had turned before. He said three Rs and hesitatingly tried three

> in this case. It worked to his satisfaction and he then drew the second side. He hesitated again, saying out loud, "What turn should I use?" The researcher said, "How many turns have you been using?" He quickly issued three right turns, then hesitated again; "How far? Oh, it must be the same as its partner!" Effortlessly, he completed his rectangle.

Even though this student had built several rectangles with horizontal and vertical sides, it was not obvious to him that the same Logo commands would work for a tilted rectangle (or, indeed, that there was such a thing as a tilted rectangle). He eventually abstracted that the opposite sides must be the same length, but he had not abstracted the measure of the turns nor even realized that all the turns were of the same measure. Thus, the *LG* environment provided him with the opportunity to analyze and reflect on the properties of a rectangle. Note also that because a computer and classroom environment had been created that encouraged exploration, this student had posed a significant challenge for himself.

We observed a similar phenomenon with second graders using standard Logo commands. The students could easily give a procedure to make the turtle draw a square with its sides vertical and horizontal. But as soon as they were asked to draw an oblique square, they had to confront their lack of abstraction of the required turns. (Like the first graders, they showed evidence of having abstracted the notion that all sides are the same length).

Often, such insights are facilitated by discussion. A different first-grade class was discussing a group of quadrilaterals drawn on the chalkboard, trying to identify the rectangles (in preparation for writing a procedure to draw rectangles). They were focusing on a nonrectangular parallelogram.

John: This one is slanted. It can't be a rectangle.

Cathy: Slanted doesn't matter. It has two long sides, here and here, the same length and two short sides, here and here, the same length [motioning to indicate pairs of opposite sides].

Eugene: But it doesn't have square corners.

Thus, the students were grappling with the properties of rectangles.

In another instance, a kindergartner explained that "rectangles need two numbers for these sides (indicating one pair of opposite sides with two hands) and these sides, but squares need just one, because every side is the same length." Such conversations reveal substantive attention to properties of figures.

Student work on another activity, "Rectangles: What Can You Draw?" (see Figure 11), demonstrates the hypothesized progression in van Hiele levels and from nonanalytical to empirical to logical thought. In this task, students are shown a variety of quadrilaterals and are asked to determine if each could or could not be drawn with a Logo rectangle procedure that takes two inputs and to explain their reasoning. Many students, even in Grades 5 and 6, often did not try the square, saying, "It's a square, not a rectangle." However, soon they found that some of their classmates succeeded in producing a square with the rectangle procedure. This led to involved and passionate discussions regarding the relationship between these two

shapes. As can be seen, a strength of this activity is that it requires students to confront the conflict between the relationships, embodied within the Logo procedures, and their own conceptions of the figures. Without working on the Logo activities and discussing them with their peers, students may have just refused to even consider the possibility of an inclusive relationship between the square and the rectangle.

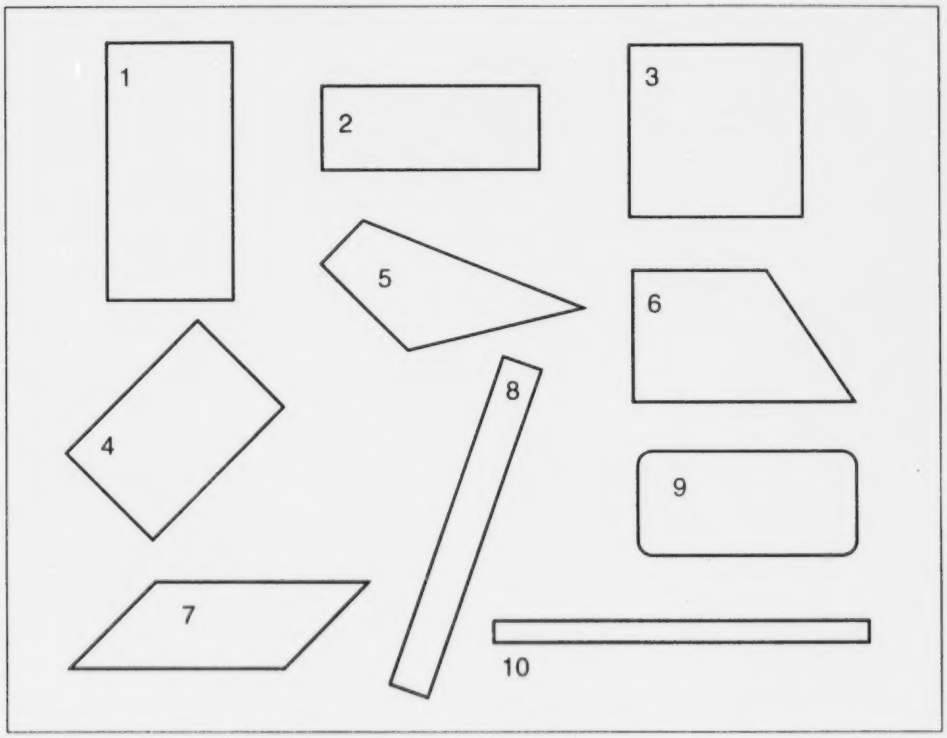


Figure 11. Figures for the "Rectangles: What Can You Draw?" activity.

Other observations support the finding that *LG* activities, including the computer environment and classroom dialogue, support students' development of higher levels of geometric thinking. After making the tilted rectangle (#4 above) by first turning the turtle, students often say that the nonrectangular parallelogram (#7 above) can also be drawn. They are quite shocked when it does not work. Many who reflect on the difficulty conclude that the nonrectangular parallelogram cannot be drawn because it does not have 90° turns. But, without the computer activity, these students would not have progressed as far as they did. That is, they would not have confronted the conceptual inconsistencies that, in turn, produced the productive reflection. The partial dialogue of a pair of sixth-grade girls, who were working on this activity, support these conclusions. They also illustrate another valuable aspect of the computer environment, providing a way for students to inde-

pendently test their mathematical ideas.

Vanessa: I don't think that you can do it. [She was then ready to go on to next problem.]
Cheryl: Yes, you can.
Vanessa: You can't because—
Cheryl: Let's try it. [The students try it on the computer and observe the result.]
Vanessa: You can't do it, because the turns are not 90 [degrees].

Interestingly, Vanessa did not feel that it was necessary to try this example. Actually, she possessed a good reason for her belief, but in the face of conflict, she decided not to share it. This girl seemed to need to have her theory validated on the computer before she was willing to publicly argue it. Many students at Vanessa's age level are just beginning to develop the ability to reason about mathematics. They need affirmation that their tentative conclusions are correct in order to build confidence in this nascent mode of thought. Computer experimentation allows them to test, refine, and validate their reasoning and provides them with the opportunity to gain that confidence. Also, as has been previously observed, this episode shows that students determine what is true or not true with their own reasoning. They do not use an adult authority to resolve conflicts. These actions further develop confidence and habits of autonomous thinking.

In some cases, Logo activities may affect the conceptualizations of very young children, even when they are engaged in noncomputer tasks. In the final interview, students were to asked identify all the rectangles in a collection of figures. At the pretest interview, most students, from kindergarten to sixth grade, confused nonrectangular parallelograms and rectangles at oblique orientations. First-grader Andrew was no exception; he included several nonrectangular parallelograms as rectangles on the pretest. After experiencing the Logo-based curriculum, however, Andrew correctly identified a rectangle at an oblique orientation and also correctly dismissed a nonrectangular parallelogram, saying: "That's not a rectangle; that's slanty."

Interviewer: Well, isn't this one [indicates oblique rectangle] slanty, too?
Andrew: It's okay. It's just turned. Look. If this pencil was a turtle, you could turn it like this [turns pencil from oblique to horizontal orientation]. Then it would be a regular rectangle.
Interviewer: But what if I did that to this one [indicates the parallelogram]?
Andrew: No, even if you turned it, it wouldn't be a rectangle. It would still have slanty sides ... [indicates nonright angle with his fingers] slanty to each other!

Such observations confirm that these students are developing higher levels of thinking. They are using Logo as a tool to test out their ideas and thus promote that development. This development is not instantaneous nor trivial, of course. We also have evidence that students just beginning work in Logo are indeed still "in transition" to Level 2. One interviewed student, Luke, who initially claimed that rectangles possess right angles, retracted that claim after noticing that several of the shapes he had selected (nonrectangular parallelograms) were inconsistent with this description. Indeed, after selecting as "rectangles" all parallelograms that were not rhombuses, Luke was asked by the interviewer why he did not choose a trapezoid.

Luke: Because these two lines are not parallel to each other and it doesn't have a right angle.

Interviewer: What are all the special things about a rectangle? You already said four sides.

Luke: A right angle, I think. Yeah, a right angle.

Interviewer: So, a rectangle has to have a right angle?

Luke: Yeah. Or four 90° angles.

Interviewer: How about [indicates a nonrectangular parallelogram]?

Luke: Well, it doesn't have to have a right angle, but the four sides are parallel. [Pause.]

Interviewer: You seem puzzled. Tell me what you are thinking.

Luke: 'Cause that looks like a rectangle, but it doesn't have any 90° angles. [Pause.] I don't think that a rectangle has to have a 90° [angle]....

This type of tension between the visual and descriptive/analytic perspectives may indicate the difficulty students have integrating their visually based recognition of figures with their emerging knowledge of the properties of geometric figures. *LG* provides situations in which such cognitive conflicts frequently emerge as well as a tool (e.g., student-defined procedures for rectangles) for investigating, and often resolving, these conflicts.

Other episodes specifically address the notion of hierarchical classification. For example, consider kindergartners Chris and Robbie. They had, as a part of the Singlekey environment, a "Shape" command to draw figures of various sizes. For example, they could type S (for Shape), receive a selection of shapes, type the first letter, (e.g., S for Square), and then receive a prompt to type a number for the length of each side. Use of these Singlekey Shape commands led to interesting discussions about squares and rectangles, as illustrated by the following dialogue:

Interviewer: Clear the screen and show me another shape.

Chris: S. I want a square. S again. I want a nine.

Interviewer: What other kinds of shapes does this thing make? [Pause.] Can you show me any other shape?

Chris: [Shows a rectangle.] S. Rectangle. R! Nine.

Interviewer: How come it didn't do it yet?

Chris: [With a hint of annoyance]: I don't know! I pushed S then R then 9.

Interviewer: It's asking you for another number. Do you need two numbers for a rectangle?

Chris: Yes [presses another number]. And another 9. [Views the screen and laughs!]

Interviewer: Now, what do the two nines mean for the rectangle?

Chris: I don't know, now! Maybe I'll name this a square rectangle!

Chris uses his terminology again on subsequent days. He was asked to draw what the turtle would draw if it were given the command S R 5 5. Chris drew a square.

Interviewer: That looks like a square.

Chris: It's both.

Interviewer: How can it be both?

Chris: 'Cause 5 and 5 will make a square.

Interviewer: But how do you know it is still a rectangle then?

Chris: 'Cause these look a little longer and these look a little shorter.

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Interviewer: [Gives him an accurate square] What about that?
Chris: A square.
Interviewer: Would it also be a rectangle, or not?
Chris: No.
Interviewer: Even though I made it with the rectangle command?
Chris: It would be a rectangle square.

Chris was later asked, "Is a square a special kind of rectangle?" He maintained his language, but revealed that a visual impression was the criterion of choice for him, as the following indicates:

Chris: A square ... if you pushed 9 and 9, you would make a square.
Interviewer: So, is a square a special kind of rectangle?
Chris: Yeah, if you pushed both numbers the same.
Interviewer: How about 10 on two sides and 9 on the other two? Would that make a square? Or a rectangle? Or both?
Chris: It's both [a square and a rectangle].
Interviewer: Is it a square?
Chris: Yes.
Interviewer: How come it's a square?
Chris: 'Cause 9 is close to 10.
Interviewer: Some people would say that it has to be exactly the same on each side to be a square. Who's right?
Chris: I think I am.

The last time this issue was discussed, Chris was drawing different-sized rectangles (a Piagetian task of drawing ever-smaller figures) and got down to a square.

Interviewer: Is it still a rectangle?
Chris: It's a square rectangle.

Another kindergartner, Robbie, begins in a similar way to Chris. The first time he uses the Shape command, he is momentarily surprised that two inputs are requested. As the following protocol shows, there is evidence that Logo is helping Robbie to think hard about the measures of the sides of rectangles, but it also engenders in Robbie a resistance to double-naming a square as a rectangle.

Asked to show a square, Robbie, with a bit of help, pressed S, S, then 1. Asked to make a bigger square, he pressed S, S, 7.

Interviewer: Now, what's 7 on that?
Robbie: These and these [indicates all sides].
Interviewer: Show me any other shape you can make.
Robbie: [Chooses a rectangle, then 9.]
Interviewer: Why does it want you to press another number?
Robbie: [Animated.] Oh! Because it's ... each ... side ... these two [holds arms up vertically, one on each side of himself] sides are the same, and these two [holds arms up horizontally, one above the other, indicating the top and bottom sides] are the same.
Interviewer: Oh. So, you said 9 for the first two sides, here and here, and now what are you going to make the other two sides?

Robbie: 1. [He does so.]

Interviewer: So, is this nine?

Robbie: Yeah.

Interviewer: So, where's the one? [Robbie indicates it.]

Interviewer: Okay, show me a square. [Robbie starts the keying sequence.]

Interviewer: Okay, does this need two numbers or only one?

Robbie: One.

Interviewer: How come the other one needs two, and this one only needs one?

Robbie: 'Cause these [holds arms up vertically, one on each side of himself] and [holds arms up horizontally, one above the other, indicating the top and bottom sides] ... all of the four sides are the same [shows all four sides].

Interviewer: Okay, then try this [rectangle 9 9]. [As the result shows on the computer screen] What will you get?

Robbie: [Doesn't answer until the drawing has been completed.] It'd make a square [smiles and squirms in his seat].

Interviewer: Now, is it a rectangle? Because we used the rectangle command?

Robbie: Yes.

Interviewer: Is it a rectangle, or a square, or both? What do you think?

Robbie: Both.

Interviewer: It's both. Can a rectangle be a square?

Robbie: Sometimes.

Interviewer: Sometimes? When is a rectangle a square?

Robbie: When you make a mistake.

Interviewer: What do you mean? What kind of mistake?

Robbie: When you do your job on the paper. And you're trying to make, and you didn't know how to make, a rectangle. [Throughout this time, points at the square on the screen, gesturing around its perimeter.] And you made it ... and you made it as a square.

Interviewer: Oh. Okay. But you made it with the rectangle command.

Robbie: Umm huh. Yes. [No real engagement with this issue here.]

Interestingly, Robbie intuitively uses equal input to the R (Rectangle) command when he wishes to draw a square. But, he is hesitant to call any figure produced by that command a square, as illustrated below.

Interviewer: Try to use the rectangle command.

Robbie: This isn't a rectangle.

Interviewer: What is it?

Robbie: A square.

Interviewer: So, what commands would you use?

Robbie: S and S and then 5.

Interviewer: Do you have to give just one number?

Robbie: Yes. Because all these sides would have to be equal.

Interviewer: What about this? What if I put in S R 5 5.

Robbie: That would be a rectangle for R.

Interviewer: Right, and then I tell it 5 and 5.

Robbie: R draws what he thinks it would be [a square].

Interviewer: What is that shape?

Robbie: A square.

Interviewer: How did that happen?

Robbie: Because if I went on the computer, and I think I put some number the same, I did a square, and I wanted a rectangle.

Interviewer: Why is that?

Robbie: I can go wrong on the rectangles. Because the rectangle is like a square, expect that squares aren't long. [Pause.] But on rectangles they are long.

Interviewer: What else do you know about a rectangle? What does a shape need to be to be a rectangle?

Robbie: All of the sides aren't equal. These two [opposite] and these two [other opposite] sides have to be equal.

Interviewer: How about 10 on two sides and 9 on the other two? Would that make a square?

Robbie: Kind of like a rectangle.

Interviewer: Would it be a square, too?

Robbie: [Pause.] I think may ... [Shakes head negatively.] It's not a square. 'Cause if you make a square, you wouldn't go 10 up, then you turn and it would be 9 this way, and turn and 10 this way. That's not a square.

As this episode indicates, the Logo microworlds proved to be evocative in generating thinking about the similarities and differences of squares and rectangles but, for these kindergartners, did not evoke classificatory responses. We next turn to observations in an intermediate-grade classroom.

After participating in the paths unit of *LG*, both Jeremy and Jonathan, fifth graders who were paired with each other, evinced concepts of rectangle and square that reflected this experience with paths, but these concepts were otherwise relatively unelaborated. For example, when Jeremy was asked in an interview, "What is a rectangle?" he replied as follows:

Jeremy: It's like kinda square but flatter like that [gestures]. It has straight lines and no curves.... It's not a straight path, because it turns. Here, here, here, and here. It goes on and on.

Interviewer: How about a square? How do the two differ?

Jeremy: A rectangle is longer. A rectangle is bigger.

Interviewer: What if I draw a square like this? [Draws a bigger square.]

Jeremy: They can be all different shapes. [Probably means sizes.] And squares have parallel lines. They're all the same.

Interviewer: This one's parallel to this one. [Pointing to adjacent sides, believing he does not mean what I mean by parallel. I am right to believe so.]

Jeremy: Yes, they could be lined up [indicates stacking lines one above the other so that the identical length is obvious].

Interviewer: Okay, how about a rectangle?

Jeremy: There's only two that are parallel, and two on the other side. [He draws another.] On a square they have to be 'cause all the sides are the same size.

Jonathan's response was similar, though he mentioned only side lengths ("two long sides and two short sides") and not "parallelism." Subsequently, both boys

wrote procedures to draw rectangles and engaged in the "Rectangles: What Can You Draw?" activity, in which they had to decide whether various figures could be drawn with a rectangle procedure that takes the lengths of the sides as input. (See Figure 11.)

Two segments of Jonathan's work with this activity are significant. The first deals with imagistic processes and thinking about properties of rectangles, the second with the rectangle/square relationship. Regarding the first, Jonathan has just successfully used the rectangle procedure to draw a tilted rectangle (labeled 4 in Figure 11) and is now reflecting on his unsuccessful attempt to make a nonrectangular parallelogram (labeled 7 in Figure 11).

Teacher: Could you use different inputs, or is it just impossible?

Jonathan: Maybe, if you used different inputs. [Jonathan types in a new initial turn. He stares at the picture of the parallelogram on the activity sheet.] No, you can't. Because the lines are slanted, instead of a rectangle going like that. [He traces a rectangle over the parallelogram.]

Teacher: Yes, but this one's slanted [indicates the tilted rectangle, labeled 4, that Jonathan had successfully drawn with the Logo procedure].

Jonathan: Yeah, but the lines are slanted. This one's still in the size [shape] of a rectangle. This one [parallelogram]—the thing's slanted. This thing [rectangle] ain't slanted. It looks slanted, but if you put it back [shows a turn by gesturing, meaning to turn it so that the sides are vertical and horizontal], it wouldn't be slanted. Anyway you move this [the parallelogram], it wouldn't be a rectangle. [Shakes his head.] So, there's no way.

During the "Rectangles: What Can You Draw?" activity, Jonathan used several imagery processes, including generating, inspecting, and transforming (Kosslyn, 1983). First, he generated an image of a rectangle, tracing it on the activity sheet. After making the initial turn and trying to choose inputs, he recognized that the relationship between adjacent sides was not consistent with the implicit definition of a rectangle in the Logo procedure. So, his developing conceptual knowledge helped him generate a robust rectangle image. Second, he inspected this image and compared it to his image of the parallelogram, noting the differences. Third, Jonathan transformed images of the figures, mentally rotating them. He noted that only the nonrectangular parallelogram is "slanted." (Of course, by "slanted," Jonathan means nonperpendicular. His emerging knowledge of the properties of figures is supported by his visually based reasoning.)

The second significant segment of Jonathan's working with this activity deals specifically with the rectangle/square relationship. Shape number 3 in Figure 11 is a square.

Jonathan: This one is not a rectangle. It's a square. It has equal sides. Do you want me to do it with this? [Indicates the screen.]

Teacher: Can you do it with your rectangle procedure?

Jonathan: No, because the sides are equal. So, that would be a "no."

Teacher: So, no matter what you tried, you couldn't make it with your rectangle procedure?

Jonathan: You couldn't, no, because the sides are equal.

Teacher: On your rectangle procedure, what does this first input stand for?
Jonathan: The 20? These sides.
Teacher: What does the 40 stand for?
Jonathan: Yeah, you could do it. If you put like 40, 40, 40, and 40. [Again, gestures.]
Teacher: Okay, try it.
Jonathan: So, that would be a square?
Teacher: Can you draw a square with your rectangle procedure? [Reads from the sheet.]
Jonathan: You could draw it, but it wouldn't be a rectangle.
Teacher: Should I put yes or no?
Jonathan: Yes ... you can draw it.

Even with prompting, Jonathan is resistant to call the square a rectangle. In his conceptualization, one can draw the square with the rectangle procedure, but that does not "make it" a rectangle.

What influence did this have on Jonathan's ideas about relationships among quadrilaterals? He appeared to understand something about the relationship. However, he had not fully constructed the hierarchical relationship, as subsequent observations revealed. The first observation involved class discussion following the "Rectangles: What Can You Draw?" activity. This episode reveals more about Jonathan's ideas and also illustrates the ideas of other members of the class and the nature of the instruction in which all the students were engaged.

The discussion centered around a homework assignment in which students looked for all possible squares in an array of squares. The identification of squares proceeded without particular event, until Jonathan outlined a rectangle and wanted to count that.

Chorus: That's a rectangle.
Davey: Well, a rectangle is a square.
Sally: Well, if you count a rectangle as a square.
Teacher: How many blocks are across the bottom?
Jonathan: Four.
Teacher: How many [gestures vertically]?
Jonathan: Three.
Teacher: So, is it a square?
Jonathan: No.
Teacher: Now, Koreen, ask that question you posed yesterday.
Koreen: Well, yesterday I thought you could do any rectangle. You could put any numbers in to make it. And then I found out if you put 50 50 it would make a square.
Teacher: Okay, so somebody give me a rule for what you found.
Donny: All squares are rectangles, but not all rectangles are squares. Only if the sides are all equal.

It was possible that Jonathan, like Davey, overgeneralized the square/rectangle relationship. (As a side note, the protocol constitutes another instance of students using Logo work to help them discuss ideas and conceptualize geometric relations.)

The class discussion continues, with the teacher emphasizing the hierarchical nature of the square/rectangle relationship.

Teacher: What are the properties of a square?

Davey: Four sides.

Teacher: Do both of them have four sides?

Leah: Yea.

Teacher: Right.

Sally: Four equal sides with 90° turns.

Teacher: Okay. Only one of those has four equal sides. Which one?

Kara: The square.

Teacher: What about the rectangle?

Abby: Well, the sides aren't equal ... the left and right sides are the same but ... they're parallel ... but the ones on the top and bottom are longer.

Teacher: That's what we want to get at. Is part of our definition of a rectangle that the vertical and horizontal lines are not the same? Is that what we're getting at? What we talked about is that the pair of horizontal lines are parallel and the pair of vertical lines are parallel. In observation, what we see when we look at what we think of commonly as a rectangle is that they are different. But Koreen proved to us that ... how many on our paper thought they could make number three [the square] by using the rectangle procedure? How?

Kristie: By using the same number.

Teacher: So, a square can be a rectangle. A special type of rectangle, but a rectangle nevertheless. But not all rectangles can go back and be squares.

Koreen: When it says on that test, find all the rectangles. Well, could we find all the squares and color them in as rectangles?

Teacher: What do you say?

Koreen: Yeah. Because you can draw 'em. So, you can do it. So, we wouldn't be marked wrong, right?

Discussion turns to a task in which different shapes are to be drawn in different colors. Here, Jonathan stated the inclusive relationship correctly (but there is no evidence that he believed that the converse was not true).

Teacher: Should a square be green or yellow?

Kara: Yellow.

Teacher: Only yellow?

Koreen: This one is a square and also a rectangle. So, what do you do with that?

Teacher: What do you say? Someone else tell me what Koreen's argument is.

Jonathan: I suggest that we should cover over the green-for-rectangle with yellow-for-square.

Teacher: Why?

Jonathan: Because a square is a rectangle.

Gerry: No! If they put squares in yellow, and they know that squares are rectangles, then why would they put the squares in yellow? They wouldn't put that if they wanted squares to be green.

The teacher misunderstood Gerry slightly at the time. Gerry said something closer to "they wouldn't have asked you to color them twice." (So, they might have said green is for "nonsquare rectangles" or something. See below.) Here is an example

of students "making meaning" out of the task, including the meaning and intentions they attribute to the people who designed the task.

Teacher: But you're saying that maybe squares aren't rectangles?

Gerry: No, they are. You could color them green for rectangles. But if they've already colored the rectangles, they wouldn't have to color the squares yellow.

Lori: I have a different question. Why can't we call squares *equilateral rectangles*?

Teacher: Could we just get rid of the name *square*?

Corgie: Well, if a little kid goes to nursery school, he might not know.

Koreen: A square classifies as a bunch of things. An equilateral rectangle doesn't classify as all the things that are square.

Teacher: Give me an example of a square that isn't an equilateral rectangle. [Then explains what "equilateral" means.]

Koreen: Well, like a diamond.

The teacher draws one and has him clarify that he means a diamond with 90° turns. Koreen still maintains that the drawing is not right.

Lori: All you have to do is turn it and it would be both a square and an equilateral rectangle in my definition.

Teacher: Can anyone help Koreen come up with a square that's not an equilateral rectangle?

No one can. They leave it for "something to think about."

The issue of "correctness" comes up once more. The teacher shows several quadrilaterals. He asks several students to identify rectangles, which they do. Koreen, unsurprisingly, points to a square.

Teacher: Why is Koreen pointing to the same figure as Donny pointed to [that is, a square]? Donny?

Donny: Because it has the same properties as a rectangle.

Teacher: Yes, and we figured that out by making a square with our rectangle procedure. A polygon is a figure that is made of straight parts … it is a closed figure; it has straight parts. It is a closed, simple figure that is made up of straight parts, and bends. Just like a triangle.

Koreen: When we're in sixth grade, and we have a math test, and it says, "Mark the X on all the rectangles," and we get the rectangles wrong because we X'd out a square, now, who should we come to? [General laughter.]

Teacher: You know that answer. You tell the teacher what?

Donny: 'Cause a square has all the properties of a rectangle.

Teacher: Right, and that should convince the teacher.

These discussions illustrate students coming to their own terms with difficult ideas. While these students are still consolidating Level 2 thinking about properties, they are challenged by Level 3 questions. It is questionable, however, whether many had constructed complete and viable knowledge about hierarchies of quadrilaterals. To explore this in more detail, consider the following excerpts from interim individual interviews with Jonathan and Jeremy:

Interviewer: Do you remember what Koreen was saying about squares and rectangles …? What would you do … tell the teacher?

Jonathan: I'd mark them all. I would say you could do it, 'cause it's just like a rectangle, it was four sides and 90° angles. The difference is, they ain't the same, all the lines [pointing].

Interviewer: What if the teacher said mark all the squares, could you mark the rectangles?

Jonathan: No.

Interviewer: How come it doesn't go both ways?

Jonathan: Maybe it could be a square. I don't really know.

Interviewer: How could you find out if that was a square or not? How could you tell?

Jonathan: Just take a guess.

Again, Jonathan states the inclusive relationship and now spontaneously provides a property-based rationale. For the converse, he responds correctly at first, but under further questioning, he is unsure of himself and abandons properties (or avoids the issue) by guessing. The transitional nature of his ideas is shown again in the end-of-the-year interview.

Interviewer: This one is not a rectangle?

Jonathan: No. It's a square.

Interviewer: And so you can't call it a rectangle.

Jonathan: Yeah, you could, because if you did that rectangle procedure, and did 40 40, it would be a square. [Jonathan spontaneously used this argument, and it was convincing to him—possibly the sole element of his thinking sufficiently convincing to break the entrenched ideas.]

Interviewer: Okay, but would you put it with these [rectangles]?

Jonathan: Yes.

As the previous interview shows, Jeremy's ideas were different. After the previously described individual and group work, Jeremy participated in the following individual interview. In contrast to the initial interview on rectangles, Jeremy's conceptualizations here are considerably closer to established mathematical ideas.

Interviewer: What is a square in your own words?

Jeremy: A square has four equal sides, four 90° angles, it can be any shape ... uh ... size. It is, in a way, an equilateral rectangle.

Interviewer: A rectangle is what, then? What's the difference between that and a square?

Jeremy: The horizontal sides are the same length and the vertical sides are the same length, but not in relation to each other.

Interviewer: Could they be the same length?

Jeremy: Yes, but usually they are different lengths.

Interviewer: What if a kid said "two long sides and two short sides," is that a good definition?

Jeremy: No, because they have right angles.

Interviewer: If we add right angles to the definition?

Jeremy: No, you could still be thinking of different things. Many different things. And no squares.

Jeremy's progress is evident. The next segment of the interview even more clearly reveals the cognitive processes Jeremy is using to establish relationships among classes of figures.

Interviewer: What do you think of when you think of quadrilateral?
Jeremy: I think of things with four sides, usually rectangles and squares.
Interviewer: What else?
Jeremy: Rhomboids.
Interviewer: What is that?
Jeremy: It's a square turned sideways.
Interviewer: Can it also be different from a square?
Jeremy: Yes. [Draws a parallelogram. The interviewer says that a rhombus must have all sides equal and shows Jeremy examples.]
Interviewer: So, are rhomboids squares?
Jeremy: No, 'cause they don't have four right angles.
Interviewer: Are squares rhomboids?
Jeremy: Yes.
Interviewer: Are ... do you know what parallelograms are, with opposite sides equal and parallel?
Jeremy: Yes.
Interviewer: Are rhomboids parallelograms?
Jeremy: Yes.
Interviewer: Every one of them?
Jeremy: [Looks up at the ceiling for quite some time.]

This raises the question of how Jeremy was answering the queries posed to this point. The following interchange suggests an answer:

Interviewer: When you look up and think about that, what are you thinking?
Jeremy: I visualize. Different rhomboid shapes.
Interviewer: Can you draw some?
Jeremy: [He does so; they are different in size, orientation, angle size.]
Interviewer: So, how do you check them out to see if they are parallelograms?
Jeremy: I couldn't think of any that weren't.
Interviewer: So, you figured they all were.
Jeremy: Yes.
Interviewer: Now, are all squares rectangles?
Jeremy: No.
Interviewer: Did you visualize?
Jeremy: [Very quickly.] No.
Interviewer: Oh, so you have a different way? How did you know that one?
Jeremy: Well, I watched *Square 1 TV.*
Interviewer: What do they say?
Jeremy: All squares are rectangles, but not all rectangles are squares.
Interviewer: How do you know they are right?
Jeremy: It says in textbooks.
Interviewer: Is there any way you could prove it mathematically?
Jeremy: Well, I could tell her the definition of a rectangle and the definition of a square. All rectangles have four sides, and the opposite sides are parallel.
Interviewer: Is that all?

Jeremy: And four right angles. And squares have all that.

Interviewer: Okay. Do you have to say four right angles? Or would it be enough to say, "at least one right angle."

Jeremy: [Looks at ceiling.] You would need that, 'cause then you could think of something like *this*—the opposite sides are parallel, and that isn't a closed shape.

Interviewer: Ah, let's say we agreed that it had to be closed, simple, quad, opposites sides equal and parallel, and at least one right angle. [Writes these down as they are mentioned.] Is that enough, or would we have to say four?

Jeremy: Yeah, this one again.

Interviewer: But that's not a quadrilateral, and it's not closed.

Jeremy: But it has four sides.

Interviewer: But it's not closed.

Jeremy: Yeah, you're right. That definition would do.

Interviewer: So, that's it.

Jeremy: Yeah, but if you said four right angles, it would be more concise.

Jeremy invokes a typical mathematical criterion, conciseness, but applies it to something like the phrasing of a statement.

Interviewer: I'm not sure what's more concise.

Jeremy: You need four right angles, because then you could have something like this.

Interviewer: Ah, a parallelogram. But that doesn't have *one* right angle.

Jeremy: [He draws some more, thinks some more.] You're right.

Thus, Jeremy appears to use a combination of processes. A basic process is visualization with mental, imagistic transformations, in which he checks the images against his schemes for various shapes. He also applies, and is influenced by, a logical argument concerning properties and statements that he has accepted, and can justify, although it is uncertain to what extent his past or present acceptance is based on his own justification.

In the end-of-the-year interview, he clearly and quickly chooses all the rectangles, including the squares, as examples of rectangles.

Interviewer: Which of these can be called rectangles?

Jeremy: All of these can [choosing all rectangles, including squares].

Interviewer: Why?

Jeremy: Because they have the properties of rectangles.

Both Jonathan and Jeremy displayed clear evidence of learning about properties of rectangles and squares, especially considering their conceptualizations at the beginning of the Shapes unit. However, Jonathan struggled throughout his work on the unit with the relationship between those classes of quadrilaterals. Logo-based experiences appear to begin to break down his certainty regarding the exclusivity of the two classes. In addition, on several occasions he states that all squares are rectangles, sometimes providing a Logo- or property-based rationale. However, at one point he appears to accept the converse of the proposition that all squares are rectangles. Later, he denies the converse but admits that he is not really sure. Finally, he does not name a square as a rectangle in the final interview until the interviewer

states, "So, you can't call it a rectangle." At that time he admits that one could and provides a Logo-based argument. Jonathan's reasoning in this situation was similar to that of the kindergartners who thought that squares and rectangles usually were separate unless the former was produced by a procedure designed to draw the latter.

In contrast, Jeremy's experiences led him to construct ideas closer to established mathematical relationships. He appears to understand the rectangle/square relationship well, giving justifications that are based on the properties of those figures (although justification that refers to authorities is also present). Further, he can apply similar reasoning to other classes of quadrilaterals, including those whose hierarchical interrelationships were not studied by the class.

Like Jonathan, Jeremy used visual processes in this reasoning. A combination of imagistic transformations and logical arguments about shapes and their properties characterized the thinking of both students. In other words, a cyclic effect was observed for both of them. Work with Logo, which involved thinking about properties and the constraints Logo code placed on the properties of figures, affected Jonathan's reasoning. His visual imagery, in turn, supported his thinking about the geometric task and the properties of figures. In general, students' concern about the interpretation of correctness and what will be accepted as correct in different situations (e.g., other classes with other teachers) is salient. Logo-based and property-based justifications seem to support students' fledging autonomy, but it is far from certain how strong their independent thinking was at this point.

Classroom observations also illustrated how Logo experiences affected students' thinking about the hierarchical classification of geometric figures. In the episode below, students are analyzing a variable square procedure (SQUARE :X) and are trying to decide how to make a variable rectangle procedure (these and other supporting data were first reported in Battista & Clements, 1992).

Teacher:	If we are going to make a rectangle, are we going to be able to use the same [variable] procedure that we used for a square?
Several students:	No.
Teacher:	Why not? Why can't we? I'm going to wait and let everybody think about that.
Kristie:	There are two longer lines on a rectangle. They are longer than a square. All the lines are not equal in a rectangle; they are in a square. So, if you think that, you can't draw a rectangle with a square procedure.
Paulie:	In the sense that the 10 or whatever you put down for the square represents all the sides, which wouldn't work because all the sides would be equal. So you'd have to make a new procedure for it.
Jennifer:	You have mentioned that opposite sides are parallel and equal. It's the same way with a square except that all sides are equal. So that the two sides that are parallel are still equal. So, a square in the sense that you're saying is a still a rectangle, but a rectangle is not a square.
Teacher:	Can we build any rectangle with the square procedure?
Jennifer:	Yes, you can.
Teacher:	Can I build a rectangle with sides of 20 and 40?
Jennifer:	No, sorry. You can't build every single rectangle with the square procedure, but you can build one rectangle with the square procedure.

Now, pairs of students move on to the "Rectangles: What Can You Draw?" activity. As they get to the square on the sheet, Jennifer says, "It's a square." Paulie illustrates his confusion over classification by saying, "A square can be a rectangle, wait. A rectangle can be a square, but a square can't be a rectangle." Jennifer starts to correct him, saying, "A square can be a rectangle." Paulie interrupts, "Oh, yeah [laughs]."

In this episode, all of these students see that the square procedure cannot be used to make rectangles. Jennifer, however, is the only student who seems capable of comprehending the mathematical perspective of classifying squares and rectangles. However, her comment, "in the sense that you're saying," suggests that she has not yet accepted this organization as her own. The next episode, in which she and her partner are interviewed, further illustrates that she has not yet adopted a mathematical organization in her classification of shapes.

Interviewer: If I typed in RECT 50 51, what would it be? [Does not press the return key, so the procedure does not yet run.]

Paulie: Probably about a square.

Jennifer: A rectangle, but it wouldn't—

Paulie: It would be a rectangle but sorta like—

Jennifer: It would be a rectangle, but it wouldn't be a perfect square. [The partners press the return key.]

Jennifer: You see, it's not a perfect square.

Paulie: [Measures the top side, or longer side, with his fingers.] It's only one step off.

Even though Paulie and Jennifer say that the 50 51 rectangle is a rectangle and not a square, their language seems to indicate their belief in such a thing as an "imperfect square"; that is, we presume, they believe in a figure that looks just like a square but does not possess the characteristic of having all sides equal. Paulie and Jennifer are clinging to an informal, rather than logical, classification system, one that still contains remnants of their visual thinking.

Finally, we examine the comments made by sixth-grader Kelly during a class discussion, in which the square/rectangle issue was raised when students tried to draw the square with the rectangle procedure. Kelly asked, "Why don't you call a rectangle a square with unequal sides?" When the teacher responded by saying, "A rectangle is a shape which has four right turns and opposite sides parallel," Kelly rejoined, "If you use your definition, then the square is a rectangle" (Lewellen, 1992). Kelly's comments, like those of fifth-grader Jennifer, which were described previously, clearly indicate an ability to follow the logic in the mathematical classification of squares and rectangles. However, neither student had made that logical network her own—each girl still clung to the personal network constructed from previous experiences. As van Hiele says, "Only if the usual [as taught in the classroom] network of relations of the third level has been accepted does the square have to be understood as belonging to the set of rhombuses. This acceptance must be voluntary; it is not possible to force a network of relations on someone" (van Hiele, 1986, p. 50). For Kelly or Jennifer to move to the next level, each must reorganize her definitions of shapes in a way that permits a total classification

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scheme to be constructed. That is, the attainment of Level 3 does not automatically result from the ability to follow and make logical deductions; the student must utilize this ability to reorganize her or his knowledge into a new network of relations. In this network, "One property can signal other properties, so definitions can be seen not merely as descriptions but as a way of logically organizing properties" (Clements & Battista, 1992b). Normally, establishing this network entails making sense of and accepting the common mathematical definitions and resulting hierarchies given in the classroom. We will return to this issue and draw additional conclusions and implications in chapter 4.

ANGLE, ANGLE MEASURE, AND TURNS

Rotations play a fundamental role in forming geometric figures in turtle geometry. Therefore, we expected that Logo experience would provide foundational experiences with rotation. Further, *LG* explicitly links turtle rotation to concepts of angle and angle measurement. Thus, we hypothesized that experience with this curriculum would facilitate the development of the geometric concepts of angle and angle size.

Pre-Post Test

Several items on the 1988–89 paper-and-pencil test assessed students' ideas about angle and angle measurement. Some items addressed the concept of rotation and the relationship between the turtle's rotation and the measurement of the resultant angle.

Identification of Angles. One open-ended question asked, "What is an angle?" We categorized responses to this question by students in all grades, K to 6. Table 16 provides percentages by category of these responses. From pretest to posttest, the *LG* responses more frequently were categorized as reflecting rotation and the intersection of two lines. There was a corresponding decrease in the no-interpretable-response category and the tilted-line category. In comparison, the control group also showed a decrease in the no-interpretable-response category (although less so than *LG*), whereas its greatest increase occurred in the intersection-of-two-lines category. Most of the other categories for the control group remained stable.

Another task on the paper-and-pencil test was designed to assess students' understanding of the concepts of angle (see Figure 12). The students were presented with a page of figures and asked to circle all the angles. One point was given for each correct response, that is, a response in which only angle vertices were circled, yielding a maximum score of 14.

Means and standard deviations are shown in Table 17. An ANOVA on the total scores for this angle identification question revealed no significant treatment effects.

Draw an Angle, Draw a Bigger Angle. Two items on the paper-and-pencil test asked students to draw an angle and then to draw a bigger angle. Each item was

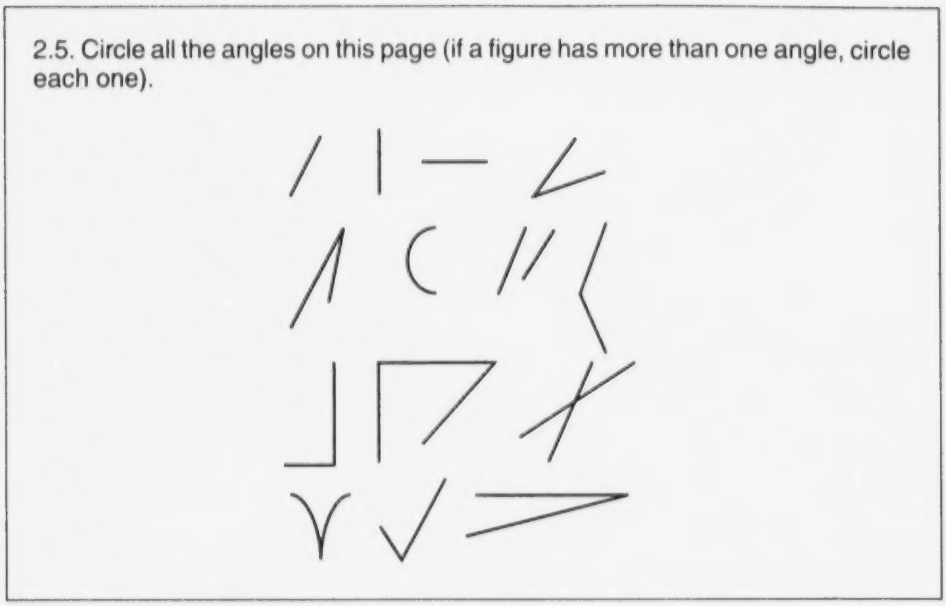


Figure 12.

worth a single point. Tables 18 and 19 provide means and standard deviations for these angle drawing items. A MANOVA on the total scores revealed several significant interactions. First was a Treatment × Time interaction (Pillais trace, $F(2, 1038) = 5.02$, $p < .01$). There was also a significant Treatment × Time × Grade interaction (Pillais trace, $F(12, 2078) = 3.34$, $p < .001$). For the Draw-an-Angle item, kindergarten and sixth-grade students improved more in *LG* than in control classrooms. For the Draw-a-Bigger-Angle item, kindergarten, third-grade, and (especially) fourth-grade students improved more in *LG* than in control classrooms.

An examination of individual responses revealed that in Grades 2–4, most errors occurred when drawing tilted lines or geometric shapes (e.g., drawing a rectangle with no sign indicating one or more of its angles). *LG* instruction had a strong effect on reducing the incidence of these errors at Grades 2–3; at Grade 4, both *LG* and control instruction reduced these errors, although control instruction did so to a lesser degree. At Grade 5, a different pattern was observed. Most of the students were able to draw angles even before instruction; instruction had the effect of changing the types of correct angles the students drew. Most fifth-grade control students shifted from nonprototypical acute angles to prototypical acute and prototypical right angles. Most fifth-grade *LG* students shifted from right angles to acute and nonprototypical obtuse angles. Shifts in Grade 6 were less pronounced; only control students shifted substantially, from prototypical right angles to prototypical acute angles.

In summary, LG students significantly outperformed control students in drawing an angle, although the difference was slight. The youngest and, to a lesser extent, the oldest *LG* students improved the most relative to control students. Similarly, *LG* students outperformed controls on drawing a larger angle, with kindergarten, third-grade, and (especially) fourth-grade *LG* students benefiting the most.

Amount of Turn (Spinner). Three spinner questions on the PP test assessed the knowledge of amount of turn for students in Grades 2–7 (one of the three similar tasks is illustrated in Figure 13). One point was given for the correct direction and turn measure (within 20°), for a maximum possible score of 3 points.

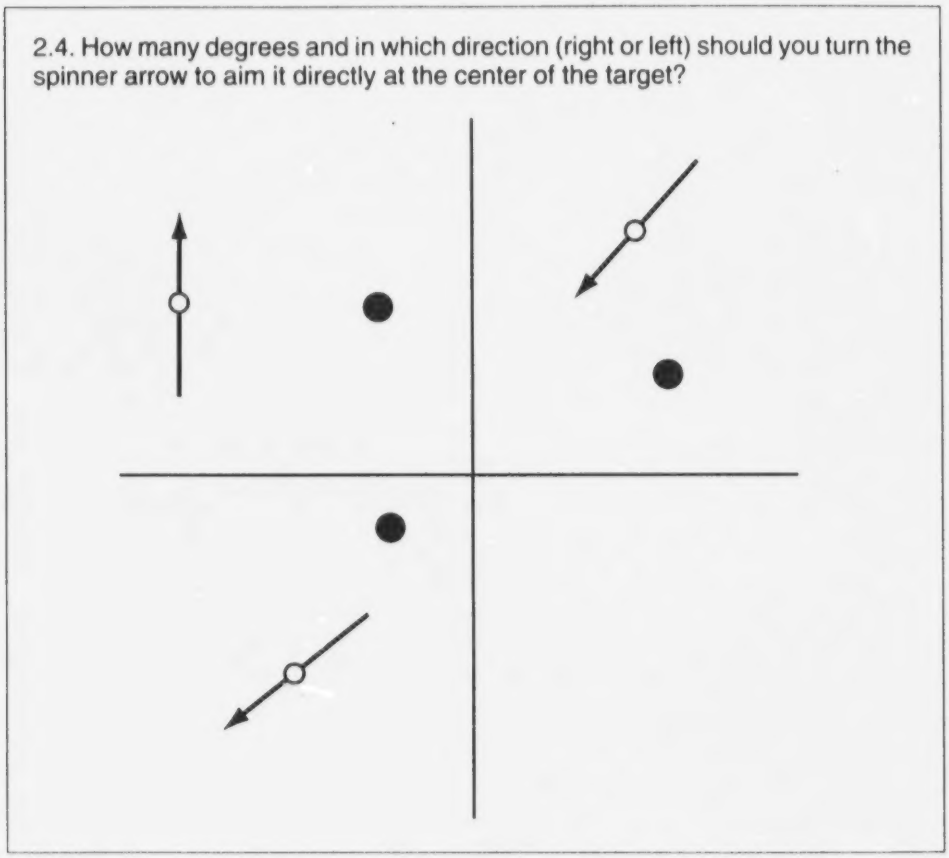

2.4. How many degrees and in which direction (right or left) should you turn the spinner arrow to aim it directly at the center of the target?

Figure 13.

Table 20 presents the means and standard deviations for this item. An ANOVA revealed a Treatment × Time interaction ($F(1, 781) = 175.97, p < .001$). *In summary, LG* students at every grade scored substantially higher than controls on questions measuring their knowledge of degrees of turn.

Angle Measure Estimation. One item assessed angle measure estimation (see Figure 14). One point was given for each correct response, for a maximum possible score of 4 points.

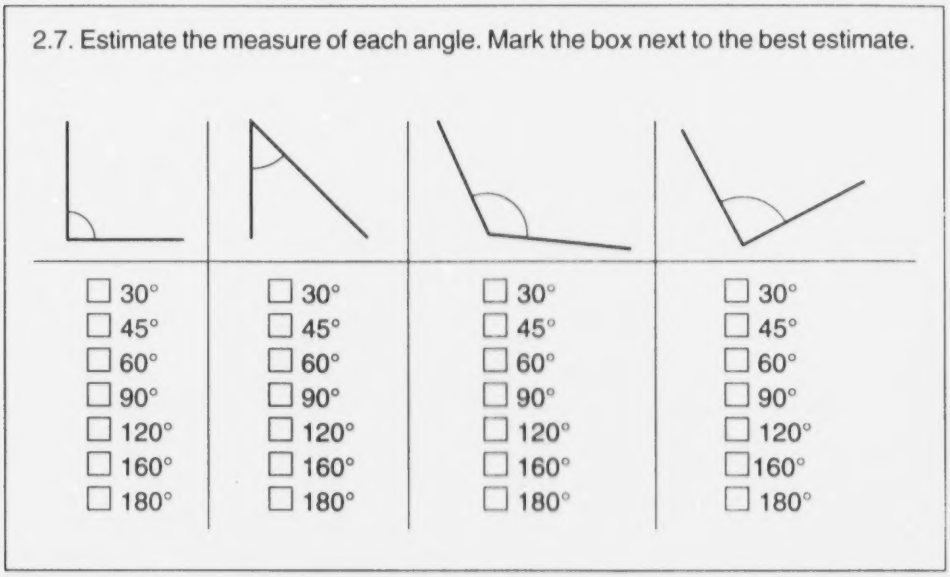


Figure 14.

Table 21 presents the means and standard deviations for these questions. An ANOVA revealed no significant Treatment effects.

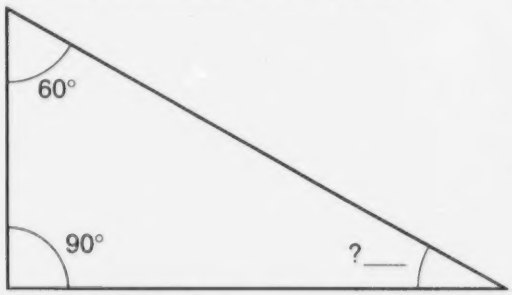
Missing Angle and Side Measures. Two items presented figures with missing angle and side measures (see Figure 15). Each response was scored as correct (1 point) or incorrect (0 points), for a maximum possible score of 7 points.

Table 22 presents the means and standard deviations for these items. An ANOVA revealed a Treatment × Time interaction ($F(1, 459) = 28.96, p < .001$). *In summary, LG* students were more capable of finding missing angle and side measures than control students.

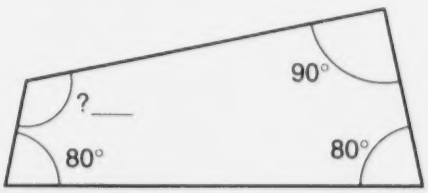
Angle Measure: Boat. Two items measured the ability of students in Grades 4, 5, and 6 to find angle measures in problem situations, both of which were scored 0 or 1. The first, involved a boat's path (Figure 16).

Table 23 presents the means and standard deviations for the Boat Amount-of-Turn item. An ANOVA revealed a significant Treatment × Time interaction ($F(1, 317) = 12.31, p < .01$). There was also a significant Treatment × Time × Grade interaction ($F(2, 317) = 3.63, p < .05$). Of the control classes, only the sixth grade made substantial gains. Further, the fourth-grade *LG* classes apparently did not make greater gains than the fourth-grade control classes.

1.8. Find the missing measures of the angles.

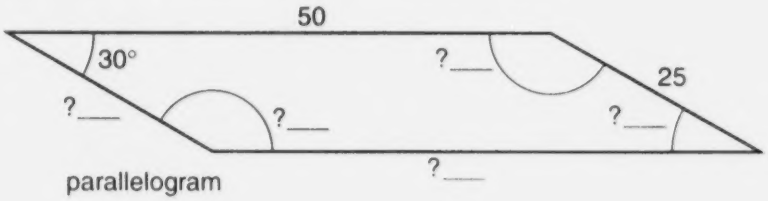


right triangle



quadrilateral

1.9. Find the missing measures of the sides and angles.



parallelogram

Figure 15.

2.12. A boat is sailing on a lake, heading toward its home. It goes forward 60 yards, turns right 80°, goes forward 152 yards, turns right 160°, and goes forward 173 yards. It is now back to its original position on the lake. How much does it have to turn to be facing toward its home again?

Answer _____

How did you get your answer?

Figure 16.

Examination of individual responses revealed that the most common erroneous response was 240°, the sum of the two given angles, 80° and 160°. Fourth graders in both groups, and fifth graders in the control group were more likely to make this error after instruction. There was a reduction in this error by the *LG* students but an increase by the control students. Other common erroneous responses, such as of 90° and 0°, appear to indicate a lack of understanding of the problem. *LG* instruction reduced the incidence of these errors, whereas control instruction increased it.

Angle Measure: Wire. The second item that measured the ability of Grades 2–6 students to find angle measures used a wire-bending situation (see Figure 17).

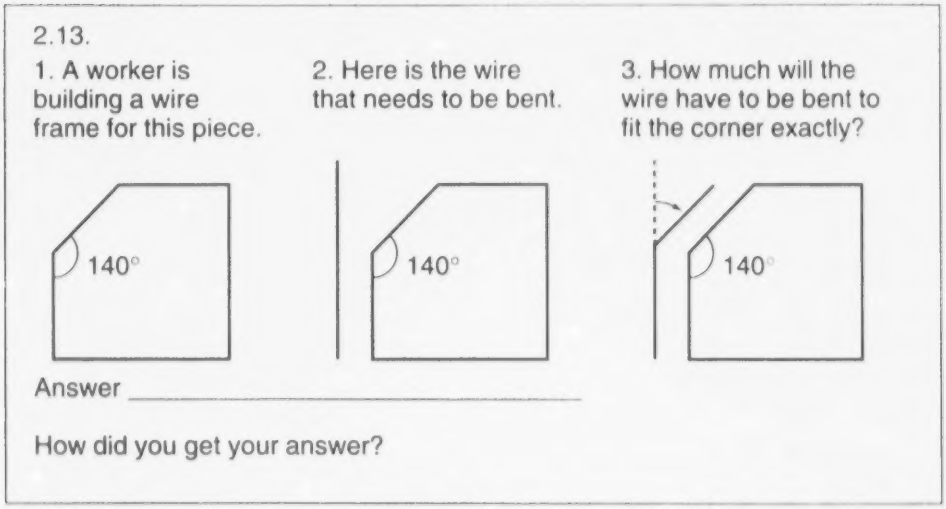

2.13.
1. A worker is building a wire frame for this piece.
2. Here is the wire that needs to be bent.
3. How much will the wire have to be bent to fit the corner exactly?
140°
140°
140°
Answer
How did you get your answer?

Figure 17.

Table 24 presents the means and standard deviations for the wire-bending item. An ANOVA revealed a significant Treatment × Time interaction ($F(1, 487) = 4.93$, $p < .05$).

A frequent erroneous response was 45°, selected by more *LG* students than control students. Even though such a response was given no partial credit, it probably indicates use of visual estimation and good measurement sense. Another common erroneous response was 140°, a misinterpretation that the question concerned the angle of the frame itself rather than its supplement. *LG* instruction reduced the incidence of this error at every grade level; control instruction reduced this error only at Grade 6.

In summary, LG students were more capable than control students of determining measures of turns in two problem situations ("boat" and "wire"). For the boat item, substantial gains relative to controls were found for fifth grade and sixth grade but

not for fourth grade. At all grades tested, *LG* students performed better than control students for the less complex wire item.

Unit Tests

Unit tests were designed to measure the specific geometric achievement goals of *LG*. They included items on identifying turns and angles and on angle measurement.

Identifying Turns and Angles. Kindergartners accurately identified the number of turns in illustrated paths (triangle, rhombus, square; 82% to 97%; *M* = 89%). Recall that first graders were only moderately competent in giving the number of turns in a square and a triangle without the benefit of a diagram (58% to 74%). Second graders were more accurate than first graders in identifying the number of angles in figures for which diagrams were provided (69% to 98%, *M* = 86%); similarly, third graders and fourth graders accurately identified the number of turns and angles in a larger set of figures (50% to 96%, *M* = 80%; 50% to 95%, *M* = 87%).

Measure of Turns and Angles. These items asked students to identify turns of a given measure, identify the measure of given turns and angles, and solve problems involving the total interior and exterior angles of polygons. About two thirds of the first graders were able to tell "how many small turtle turns (30° in the latter half of their work with *LG*'s Singlekey) make a quarter turn (90°)" (62% to 77%, *M* = 69%). Third graders performed poorly and fourth graders moderately when asked to select the number of degrees for angles of measure 30°, 45°, 90°, and 120° (given those measures, and 60° and 150°, as choices; 17% to 100%, *M* = 57%; 43% to 100%, *M* = 71%). The 90° angles were far more likely to be correctly identified by both groups. In a similar vein, fourth graders successfully identified the right angles in polygons (only completely correct responses received credit; 60% to 90%, *M* = 78%). About 95% identified all the right angles in a square with a horizontal base; 89% did so in a square with sides at a 45° orientation, with 2.5% identifying no angles and 1.7% identifying three of the four angles. Approximately 81% identified the right angle in a right isosceles triangle (with the right angle at the top), 10% identified all angles (it is unknown how many did not process the adjective "right" in the directions), 5% identified the two acute angles, and 4% identified one right and one acute angle.

Third graders were also asked to give the total amount the turtle turns when it makes a square. The range of two classrooms indicates that mastery of the idea is possible but must be facilitated by the teacher (9% to 88%, *M* = 48%). The lowest performance was from students in one classroom, most of whom answered "4" (i.e., the number of turns rather than the total amount turned). It may be that this teacher inadequately emphasized amounts of turn or that the students misunderstood the question because of poor reading ability or a misinterpretation by the teacher within the test situation.

Fourth through sixth graders were given the measure of two angles of a triangle and asked to find the measure of the third angle. Performance was low for fourth

graders, who were provided a figure (50% to 60%, $M = 54\%$), and also for fifth and sixth graders, who were not given a figure (39% to 63%, $M = 49\%$).

Fifth and sixth graders were asked several additional questions about angle-measure relationships in regular polygons, including exterior angle measures and sums of interior angle measures ($M = 69\%$). Unsurprisingly, students performed much better on questions about squares (85% to 97%) than on questions about triangles (57% to 64%) and pentagons (28% to 48%). Students scored higher on questions involving the total amount of turtle-turning (63% for the triangle to 92% for the square) than on questions involving the sum of the measures of interior angles (64% for the triangle, 85% for the square, 48% for the pentagon). However, because directing the Logo turtle emphasizes only the former, the lack of large differences indicates that *LG* helped students construct both of these two constructs.

Interview

In individual interviews, students were asked to draw angles and then "angles that are different," to describe the differences between their drawings of angles, to circle the biggest angle given pairs of figures, and to solve problems involving the amount of turn of a robot.

Drawing Angles. Students were asked to "Draw an angle. Now draw another angle that is different from your first angle in some way...." Table 25 describes the students' first drawn angle; note that only two segments intersecting at an endpoint were scored as correct. For the *LG* groups, there was a decline in incorrect responses from the pretest to the posttest and an increase in correct responses. Note that the *LG* groups scored better than the control group. Also, for the correct angles drawn by the *LG* group, more than two thirds were nonprototypical, with only one third being prototypical (e.g., right angle, "facing" right, horizontal base). The reverse was the case for the control group. Thus, there are two differences between the *LG* group and the control group. First, the *LG* students drew more correct angles. Second, the *LG* students' increased awareness of the angle concept was richer in that they were much more likely to draw nonprototypical examples of angles.

Students' Descriptions of Differences between Their Angle Drawings. The second task asked students to draw another angle that was different in some way from the angle they drew in the first task and to explain how their new angle differed from the one before it. Students' responses were categorized. Table 26 shows the percentage of students who gave an explanation in each category for at least one comparison and which was consistent with their drawings at least some of the time. For the *LG* group, there were decreases from pretest to posttest in responses indicating a total lack of knowledge of the concept of angle, increases in responses indicating an intuitive understanding of angle, and increases in responses indicating a knowledge of the measure of angles or the classification of angles (by measure). The number of students that thought about angle size in terms of tilt, heading, or turning along the path remained basically constant, with tilt declining and the more sophisticated heading increasing at two of the three *LG* sites. The control group

had equal or greater posttest scores in the "formal" category, perhaps reflecting their curriculum's emphasis on classifying angles by measure.

Angle Measure. To assess students' understanding of angle measure, they were asked to circle the biggest angle in each of the pairs of angles and in the triangle shown in Figure 18.

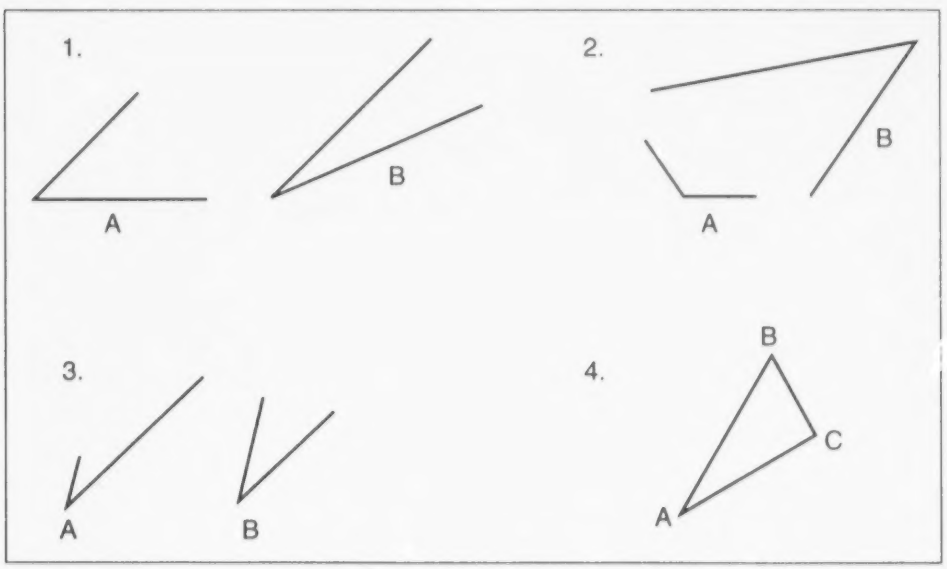


Figure 18.

The total number correct for all four items was analyzed. *LG* students scored significantly higher in judging the size of angles ($F(1, 211) = 14.01$, $p = .000$). The mean correct (out of four) was 2.16 for the *LG* group and 1.62 for the control group. For 1988–89, there was also a significant effect by Grade ($F(6, 211) = 43.02$, $p = .000$), Time ($F(2, 200) = 55.27$, $p = .000$), and Time × Grade interaction ($F(12, 200) = 4.78$, $p = .000$). In addition, for 1987–88, there was also a significant effect by Grade ($F(7, 70) = 26.22$, $p = .000$), Time ($F(2, 140) = 29.45$, $p = .000$), and Time × Grade interaction ($F(14, 140) = 2.86$, $p = .001$). Table 27 shows the angle scores by grade. As can be seen from the table, scores increased little in Grades K and 1. However, there were notable increases in the upper grades. It should be noted that the term *angle* was not dealt with explicitly in the curriculum for Grades K and 1.

The frequencies with which students chose the angles in the pairs and the triangle are given in Table 28. The number of *LG* students choosing the figure with the longer sides (choice B) for pairs 1 and 2 declined. The number of *LG* students who said that the angles were the same size in pair 3 (the correct choice) increased. For the triangle, the number of *LG* students choosing the correct angle, C, increased,

whereas the number of students who circled a side of the triangle or the whole triangle decreased.

As can be seen in Table 29, the percentage of students who correctly chose angle A in pair 2 increased from pretest to posttest, with most of the increase occurring in Grades 3–6. On the posttest, over two thirds of the students in Grades 4–6 were not confused by length of sides when judging angle size.

After each choice of a larger angle, students were asked why they made their choice. Table 30 shows the percentage of students who gave an explanation in the given category for at least one of the four items and which was consistent with the figures at least some of the time. The number of *LG* students who made choices that showed no understanding of angle size decreased. The number of *LG* students whose explanation showed an intuitive understanding of the angle concept increased at Site 2 in 1988–89 and at Site 1 in 1987–88 but decreased at Site 1 in 1988–89. Explanations that showed a formal knowledge of angle measure increased, most for the Site 1 students in 1988–89. Most of the increase in the formal category came from increases in Grades 4, 5, and 6.

Turning. Students were asked two questions concerning turning. The first question was, "A robot turns right 90 degrees (a right angle) every time it turns. How many turns must the robot make before it is facing in the same direction as it was when it started?" The second question was the same as the first, except now the robot turned 30° each time. For 1988–89, the *LG* group scored significantly higher than the control group on both items; 95% versus 74% correct ($F(1, 169) = 12.16$, $p < .001$) for the 90° problem and 61% versus 35% correct ($F(1, 167) = 11.33$, $p < .001$) for the 30° problem.

Classroom Observations

Classroom observations confirm that *LG* activities and teacher-mediation utilized in the *LG* project are critical in helping students build more abstract geometric notions from their initial intuitions about paths, including turn or angle measure. For instance, many young students in our Singlekey environment believed—despite having observed their teacher demonstrating the commands—that pressing L (for "LEFT") would command the turtle to move toward the left regardless of its heading. They had not yet constructed the idea of two qualitatively different kinds of "moves" the turtle makes: changes in position (FORWARD and BACK) and changes in heading (RIGHT and LEFT turns). Two kindergarten girls were videotaped as they began to construct this idea. The turtle was facing "down" (i.e., at a 180° heading).

Selene: Let's make it go left.
Darlene: Left is this way [points to his left].
Selene: [Presses L.]
Darlene: Ohhh [in a disappointed tone].
Teacher: What happened? What did the turtle do?
Selene: It ... turned left.

Darlene: But I want him to go left!

Teacher: Which way is the turtle heading?

Darlene: Oh, yeah, turn him and then go forward!

The teacher's questions helped the pair to reflect on the results of their actions and to attend to important features of the problem. He did not try to directly tell the children what to do.

Later in the week, the girls seemed to have firmly constructed the idea of turn along a path. The following exchange was typical.

Selene: He's facing this way, yeah. And we want him to go ... [turning her own head so that she is facing the same way as the turtle and then motioning with her hand] ... right!

Darlene: So, that would be ... R. [Taps her head repeatedly and emphatically.]

Selene: [Types R.] Now, F for forward!

The act of turning their heads to align themselves with the turtle and then motioning may be a sign that students really do "see" a change in direction as a result of a turning motion. It is important to note that the students did not construct this critical and surprisingly difficult idea by watching the teacher's demonstrations or by hearing his explanations. They used Logo as a tool to experiment with their own ideas. The teacher only helped them reflect on their experiments.

At the conclusion of a maze-like activity (to get from one point on the screen to another), the teacher had several kindergarten children show their work. He asked questions such as: At what point(s) in your path was the turtle turning? What can you tell me about the path you just made? How many straight parts does the path have? How many bends? Can you make a path with fewer straight parts? Fewer bends? Were your paths simple paths?

Student: Ours are all simple. But nobody's is a closed path.

Teacher: Why is that? How do you know for sure?

Student: Because the turtle always starts here and ends in a different place, over here. No matter where you go, that would never make a closed path!

We have investigated whether the path perspective leads to a confusion between measures of the amount of turn along the path and angle measure. Our observations indicate that under proper instructional conditions, this need not be so, even for young students. One first-grade boy was asked to draw an angle, then a "bigger angle." He sketched an angle of approximately 30°, then a right angle (90°). Spontaneously, he explained that the former was a "five turn angle" (i.e., five "R" turns of 30° each) and the latter a "three." Finally, he drew angles in which the measure of the turn along the path diminished from 150° "down to" 0°, explaining that "as you lose turns, it gets to be a bigger angle." On prompting, he explained that "lose turns" meant "going from five rights down to zero." This indicates that notions of angle, rotation, and their measures have been positively affected by work in the *LG*.

In summary, there is support from these observations for the notion that *LG* assists students in building on their initial intuitions about paths to construct more math-

ematically sophisticated concepts of geometric objects, including angle, rotation, and their measure. These are quite positive results, especially considering that these notions often cause definite problems for students. "It seems students have as much difficulty with angles as they have with any other geometric concept. Turns or rotations are natural for young children ... this dynamic way of working with angles should be started early and carried throughout the students' work" (Hoffer, 1988, p. 251). The data presented here confirm these statements and suggest that *LG* is an effective approach to implementing the recommendations to include dynamic work with angles in classrooms.

PATHS

Logo Geometry took a path perspective to elementary geometry. Therefore, we expected that *LG* would facilitate the development of path concepts. In this section, we present the small number of tasks on the PP and unit tests that assessed students' ideas about paths.

Pre-Post Test

The PP test included two types of items on paths. The first asked students to draw paths with certain properties. The second asked students for verbal descriptions of shapes to ascertain if and how they would incorporate a path perspective.

Draw a Path. Three items presented descriptions of paths and asked students to draw the paths (see Figure 19). One-half point was given if the path drawn was missing only one of the required characteristics; a full point was given for a correct response.

Table 31 presents the means and standard deviations for the paths questions. An ANOVA on the total scores for those questions administered to all students (Items

2.7. Draw one path that
has **4 straight parts**,
has **4 bends (angles)**, and
is **closed**.

2.8. Draw one path that
has **3 straight parts**,
has **2 bends (angles)**, and
is **not closed**.

2.9. Draw one path that
is **closed**,
but **not simple**.

Figure 19.

2.7 and 2.8) revealed a Treatment × Time interaction ($F(1, 1038) = 138.20$, $p < .001$). As expected, *LG* students performed significantly better.

There was also a significant Treatment × Time × Grade interaction ($F(3, 488) = 4.50$, $p < .01$). Certain control group classes seemed to make smaller gains than others. Gains for the younger *LG* classes tended to be greater than those for the older classes.

In summary, LG students outperformed control students on these path items. There was some indication that younger *LG* students made greater gains in this domain than older *LG* students, but classes at all grades outperformed control classes.

Communication: The Telephone. Item 1.10, administered to all students in Grade 2 and higher, stated, "Pretend you are talking on the telephone to someone who has never seen a triangle. Write down what you would tell this person to help them make a triangle?" [Grades K–1 students were asked, "What would you tell this person ...?"]. Students' responses were categorized. Table 32 presents the percentages by category for this item. Differences were small but indicated that *LG* students were more likely to provide complete, correct descriptions of the process of creating a triangle and that they made greater increases in the use of sophisticated strategies ("Logo-like commands" or "complex path or components together with Logo").

Unit Tests

On the unit tests measuring *LG* students' achievement of *LG* goals, students were asked to identify or draw paths with specific properties. Kindergartners identified paths with various properties, including being closed, straight, and having a given number of turns (50% to 100%; $M = 79\%$). First graders to fourth graders drew paths with such properties (first: 77% to 100%, $M = 95\%$, with a single property for each question; 2nd: 71% to 100%, $M = 94\%$, with a single property; also 47% to 87%, $M = 72\%$, with multiple properties; 3rd: 56% to 100%, $M = 87\%$, with a single property; 4th: 55% to 100%, $M = 87\%$, with a single property). Second graders also identified certain components of a path (76% to 100%, $M = 90\%$). Fifth and sixth graders decided whether to attribute each of three properties—closed, straight, and simple—to nine paths, with credit given only if all three properties were correctly attributed to the path (69% to 78%, $M = 72\%$). Of the single properties, straight and closed tended to be mastered by students, whereas the concepts of possessing a given number of bends, and especially possessing "curvy parts," tended to be more difficult. Thus, students were moderately successful in learning paths concepts.

SYMMETRY

The important geometry topic of symmetry is treated in two distinct ways in *LG*. First, computer tools allow students to make symmetric paths (MIRROR). One

turtle would follow the students' commands precisely, the other would create a dynamic "mirror image" by turning in the opposite direction (e.g., left 45 instead of right 45). Second, computer tools allow students to perform flips; students could then flip shapes to check if they were symmetric. Therefore, we expected that *LG* would facilitate the development of symmetry concepts.

Pre-Post Test

Draw Symmetry Lines. Item 1.5 asked students to draw in all lines of symmetry for various figures. Students at every grade level were presented with the first four figures in Figure 20. The fifth figure was presented only to students in Grade 3 and above. Each correct line of symmetry was awarded one point; a point was subtracted for each line drawn that was not a line of symmetry, such as any line drawn for the parallelogram. The maximum possible score was 8 points.

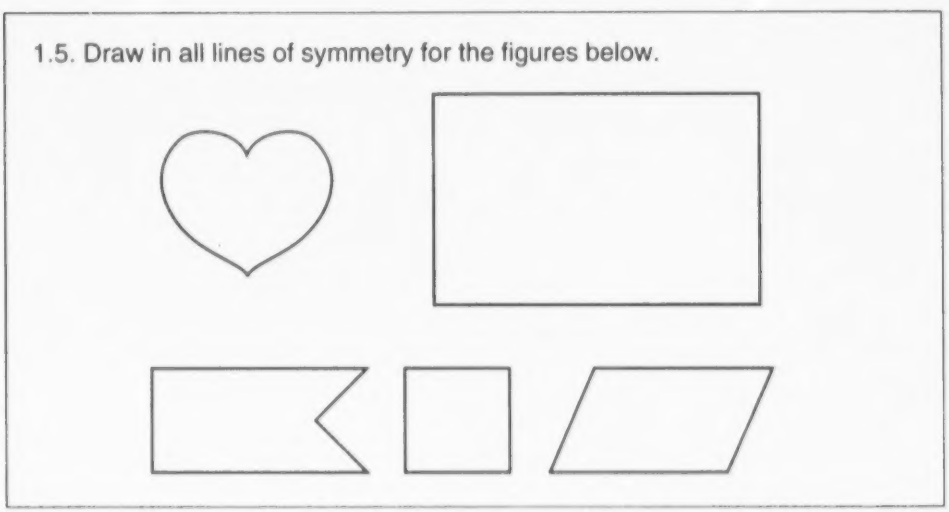


Figure 20.

Table 33 presents the means and standard deviations for the symmetry line-drawing questions. An ANOVA on the total score for the first four figures revealed a Treatment × Time interaction ($F(1, 1021) = 116.12$, $p < .001$). An ANOVA on the total scores for the draw-symmetry-lines questions also revealed a Treatment × Time interaction ($F(1, 609) = 30.98$, $p < .001$). The lower scores for the composite with more items makes sense considering that any lines drawn on the parallelogram would be incorrect, lowering the score; perfect performance on that figure adds zero to the score.

Table 34 shows the percentage of correct and incorrect symmetry lines drawn by each group. Differences between groups varied on the incorrect lines. *LG*

students improved slightly more on the item (pentagon) that had only one salient line of symmetry. Both groups increased in the errors for the rectangle item (identifying diagonals), with the increase of the *LG* students being greater. Both groups' errors on the parallelogram decreased, with the control group decreasing more. Thus, there is evidence that *LG* students overgeneralize with regard to diagonal lines of symmetry and, to a lesser extent, lines of symmetry for figures that have only rotational symmetry.

With regard to correct lines of symmetry drawn, there were clear differences between the two groups. *LG* students consistently improved more than control students, with most of the former finding all correct symmetry lines (e.g., about two thirds of *LG* students, compared to one third of the comparison students, finding all four lines of symmetry for the square).

In summary, LG students scored significantly higher than control students on drawing all the symmetry lines for given figures. *LG* students did not differ substantially from control students in the number of incorrect symmetry lines drawn, but there was a clear, positive effect of *LG* regarding correct lines of symmetry drawn. This may be due to *LG*'s inclusion of MIRROR and flip procedures.

Draw the Other Half. As shown in Figure 21, Item 1.6 asked students to draw the other half of a figure to produce a symmetric figure. One point was given if the drawing was symmetric; that is, if measures of lengths and angles were approximately correct. One-half point was given if the figure was not symmetric but the drawn half had the same number of segments on the opposite side of the line of symmetry.

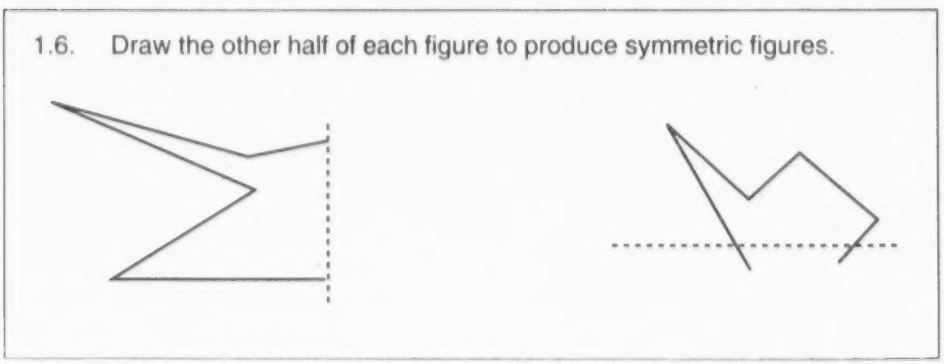


Figure 21.

Table 35 presents the means and standard deviations for the total scores for the Daw-the-Other-Half (to make a symmetric figure) tasks. An ANOVA revealed a Treatment × Time interaction ($F(1, 609) = 30.98$, $p < .001$). The largest gain by the *LG* group, compared to the control, was in the number of students receiving full credit (23% vs. 15% and 14% vs. 4% increases for the two questions).

In summary, LG students scored significantly higher than control students on drawing the "other half" of a figure to create a symmetric figure. There was a tendency for these effects to be particularly strong for young (kindergarten) *LG* students. In addition, the effects were largely due to the greater number of *LG* students drawing completely correct figures.

Unit Tests

Identification of Symmetry Lines. Students were asked to identify whether a dotted line drawn on various figures was a line of symmetry. The students' average correctness was generally high for each grade tested; average correctness scores were: Grade K (60% to 100%, $M = 95\%$), Grade 3 (23% to 100%, $M = 89\%$), Grade 4 (44% to 100%, $M = 93\%$), and Grades 5–6 (49% to 100%, $M = 93\%$). Across all grades, the most difficult item was a concave pentagon (see Figure 22) and a parallelogram, where in each case the line was not a line of symmetry.

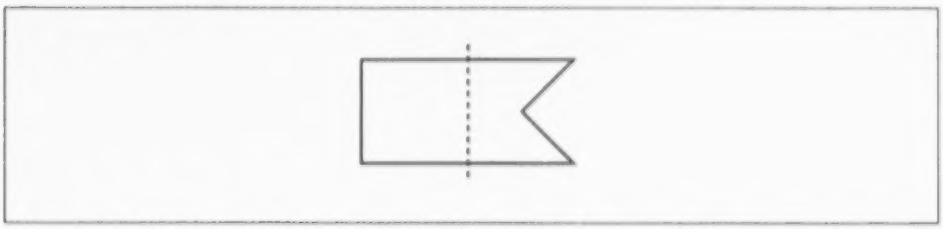


Figure 22.

In another problem, students were asked to draw all the lines of symmetry onto given figures. Two points were given if all correct lines were drawn. One point was given if half or more correct lines and no incorrect lines were drawn or if all correct lines and some incorrect lines were drawn. Average correct scores were moderately high: Grade 1 (0.78 to 1.95, $M = 1.55$), Grade 2 (1.10 to 1.61, $M = 1.46$), Grade 4 (0.58 to 2.00, $M = 1.62$), and Grades 5–6 (0.80 to 1.98, $M = 1.64$). The most difficult shape administered to students in all grades was a square; an equilateral triangle and parallelogram, only administered to Grades 4–6, were even more difficult than the square. Certain figures whose lack of symmetry was visually salient (e.g., an obtuse triangle administered to Grades 4–6) were easiest.

Students in all grades were asked to draw the "other half" of one or more given figures to create a symmetric figure. The total possible score was 2. Means were as follows: Grade K ($M = 0.68$), Grade 1 ($M = 1.14$), Grade 2 ($M = 1.42$), Grade 3 (0.57 to 1.68, $M = 1.00$), Grade 4 (1.28 to 1.83, $M = 1.54$), and Grades 5–6 (1.76 to 1.93, $M = 1.82$). These results support the claim that *LG* students performed well on symmetry items.

CONGRUENCE

Congruence was not a major focus of the curriculum or assessment. Only one activity in the *LG* curriculum emphasized congruence: Students used geometric motions to determine congruence or lack of congruence in pairs of figures. Congruence is an important geometric topic, however, and thus some assessment was in order.

Pre-Post Test

Same Shape and Same Size Item. For six pairs of figures, students were asked to state whether the figures were the same size and shape, as illustrated below. One point was given for each correct identification for each size and shape. The students were then to answer the question "How would you show someone that you are right?" Justifications were initially coded. Then, 1 point was given for justifications that pertained to the item and referred to mathematical notions (e.g., geometric motions, measurement, and congruence were given 1 point each; responses in the "look like," "they are the same shape," or "because I said so" categories were scored 0). Thus, the total possible score for identification was 12 points and for justification, 6 points. In Figure 23, the full text is reproduced only for the first pair; on the actual test, this text was repeated following each pair of figures.

Table 36 presents the means and standard deviations for the questions that dealt with congruence. A MANOVA on the total scores for the congruency questions, identification (i.e., correctly identifying figures as congruent or not) and justification ("How would you show someone that you are right?"), revealed several significant interactions. First was a Treatment × Time interaction (Pillais trace, $F(2, 1012) = 50.52$, $p < .001$). Effects were more pronounced for the justification score; for each question, *LG* students outperformed control students on the posttest. The largest difference was for those questions with congruent pairs of figures (the mittens and cone). For identification, only the cone question differentiated between the *LG* and control groups.

There was also a significant Treatment × Time × Grade interaction on the congruence questions (Pillais trace, $F(12, 2026) = 3.22$, $p < .001$). Only the justification score contributed significantly to this effect. The results of two grade levels on the justification score appear noteworthy. First, the mean score of the sixth-grade control students declined. Second, all third grades had low scores on the pretest but made significant gains on the posttest, especially those in *LG*. The low pretest score stems from one *LG* and one control (cohort) third-grade classroom, whose teachers agreed to tell students to mark a "?" if they were not sure of the answer.

In summary, LG students performed better than control students on congruence items, not in identifying whether pairs of figures were congruent so much as in justifying their answers.

1.4. For each pairs of figures, tell whether they are the same size and shape.

Same shape? ☐ **Yes** ☐ **No** **Same size?** ☐ **Yes** ☐ **No**

How would you show someone that you are right? _____

(The same format was repeated for each pair of figures below.)

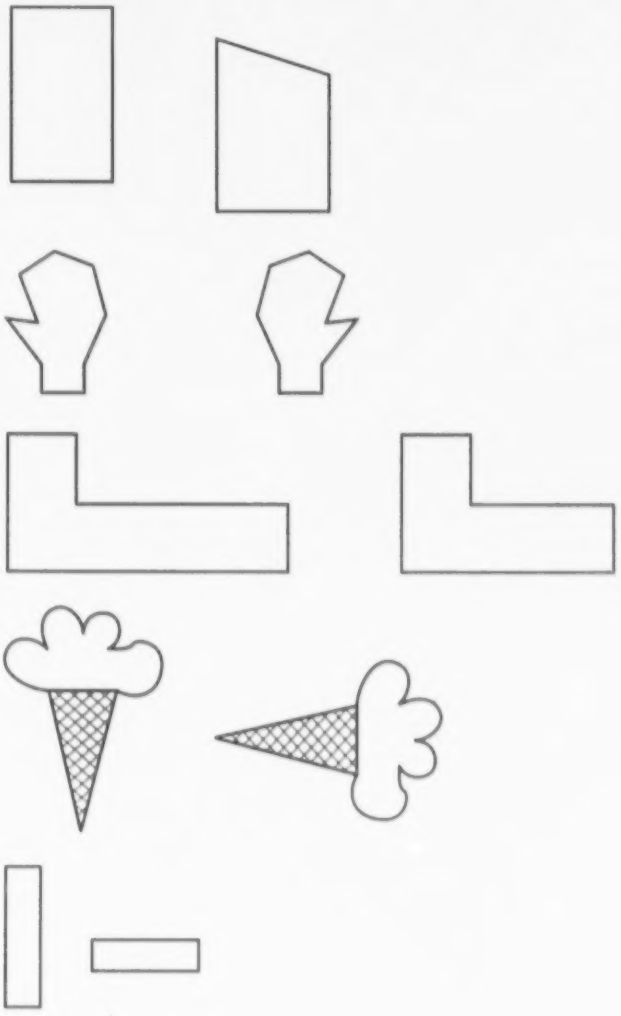


Figure 23.

Unit Tests

Similar items were presented on the unit tests. These tests indicated a uniformly high ability to identify pairs of figures as congruent or not in all grades: Grade K (M = 93% to 96%), Grade 1 (M = 92%), Grade 2 (M = 94%), Grade 3 (M = 91% and 96%), Grade 4 (M = 97%), and Grades 5–6 (M = 96%). Students scored somewhat lower for five figures as follows: block "L" hexagons, one of which was rotated 180° (administered only in Grades K–1); a pentagon and a hexagon (Grade 1 only); concave pentagons, one of which was rotated 90° (Grades 2–3); an isosceles and a right trapezoid (one "upside-down"; Grade 3 only); and squares, one of which was rotated 45° (Grades 3–6). Note that accuracy rose to about 90% across items at the higher grades; our comments about these items being the most difficult were based on within-grade comparisons and are relative to the overall high performance.

GEOMETRIC MOTIONS

Logo Geometry included several activities on geometric motions. But no items on the PP test directly assessed students' knowledge of motions, because not all control classrooms taught these competencies separately. (We did conduct a separate experimental study on this topic in which a comparison group was taught these concepts; this is described in a succeeding section.) However, several unit tests and interview items for geometric motions are relevant.

Unit Tests

Kindergarten and first-grade students identified motions to move a train engine into the same orientation as another train engine (on a 2-point scoring system, M = 1.58 and 1.22, respectively). Combinations of motions were, unsurprisingly, more difficult than individual motions (Grade 1 was administered more combinations-of-motions questions; the lowest score was on a question involving all three motions, slide, flip, and turn). An item, in which motions performed on a block "L" hexagon were to be identified (along with turn centers and flip lines), was administered to all other grades (an example problem is illustrated in Figure 24).

Accuracy was 74%, 69%, 83%, and 91%, for Grades 2, 3, 4, and 5/6 respectively (one third-grade teacher mistakenly did not have students attempt to identify turn centers and flip lines, artificially deflating the third-grade mean). Questions about a turn center not located on the figures, about a turn center located on the figures, and about a flip line for figures separated by six units were the most difficult. The most difficult motion to identify involved the item illustrated in Figure 24, a 180° rotation with a turn center not located on the figures. These difficulties diminished as the grade level increased.

Students in Grades 5 and 6 were administered two additional items. The first item asked them to draw an image given a preimage and a motion (see Figure 25).

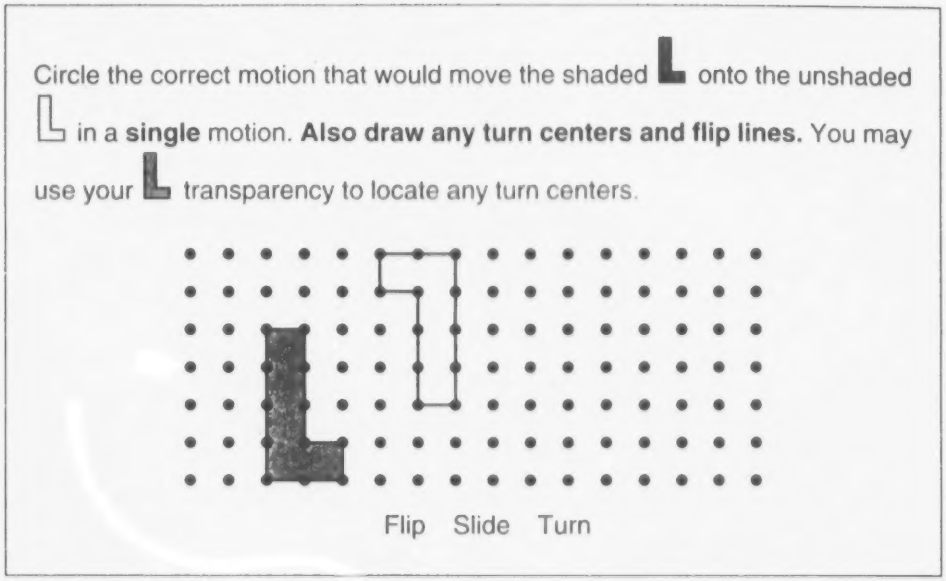

Circle the correct motion that would move the shaded L onto the unshaded L in a **single** motion. **Also draw any turn centers and flip lines.** You may use your L transparency to locate any turn centers.
Flip Slide Turn

Figure 24.

Accuracy was 76%. Turns were more difficult than the slide or flip, and a right turn with a turn center located at the top of the figure, illustrated below, was substantially more difficult than a left turn with a turn center located at the bottom of the figure.

The second item assessing geometric motions tested Grade 5–6 students' knowledge of the relationships between motions and congruence. Accuracy was moderate

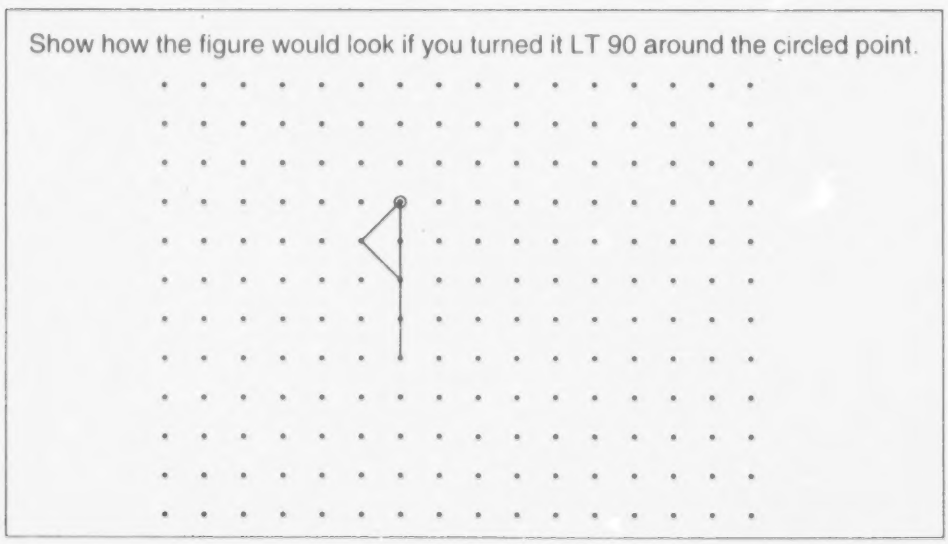


Figure 25.

(e.g., 67% to 80% for the three items administered both years). Most students appeared to understand the basic relationships (e.g., 76% accuracy on "if two figures are congruent, there is a sequence of *flips* that will move one onto the other"); the most difficult items were those for which students erroneously agreed that "if two figures are congruent, there is a sequence of slides and turns that will move one onto the other." Thus, most students understood that motions can determine congruence (recall that *LG* asked students to use procedures that performed geometric motions to determine congruence) and that for congruent figures, a sequence of motions can be found to move one onto the other. However, about half of the students may not have understood that slides and turns are not sufficient for this purpose. This was not explicitly discussed in the curriculum.

Interview

Motions van Hiele Level Task. Students' van Hiele levels for geometric motions were assessed with the task, "Moving the Triangle," described in Figure 26. To obtain a student's motions level, the following steps were involved. First, the student's explanation for his or her use of the flip, slide, and turn motions was coded. Next, the codes were classified into van Hiele levels. Finally, the average van Hiele level for the three motions was calculated.

"You and your friend are playing a game on the telephone. Here is what your friend sees. [The student was shown a larger version of the diagram below without the unshaded triangle.] Here is what you see. [The student was shown a larger version of the following diagram.] You are to tell him/her **exactly** how to move his/her triangle [gesture to the shaded one] so that it fits onto this triangle [gesture to the unshaded one] exactly. Remember, you are talking on the telephone, so your friend cannot see you or your picture. You can use this triangle piece [a cutout triangle is provided] to help you."

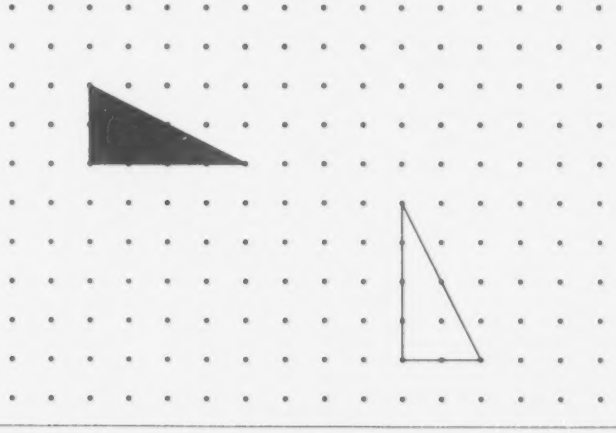


Figure 26.

The motions van Hiele level for the *LG* group (measured only in 1988–89) increased significantly from pretest to posttest ($F(1,83) = 67.22$, $p < .001$), with a significant Site × Time interaction ($F(1,83) = 8.65$, $p = .004$), with the *LG* group in Site 1 increasing more than the *LG* group in Site 2 (*LG*: Site 1, .57 to 1.16; Site 2, .67 to .95). On the posttest, there was also a significant treatment effect in favor of the *LG* group ($F(1,175) = 71.17$, $p < .001$; *LG*, Site 1, 1.14, Site 2, .82; Control, Site 1, .60, Site 2, .57).[11]

Shapes versus Motions Levels. We hypothesized that the students' motions level on the preinterview would be below their shapes level because they had less school experience studying geometric motions. Overall, about 98% of the students had a motions level at or below their shapes level on the pretest, and about 80% of the students had a motions level at or below their shapes level at the posttest. These data support the notion that students' van Hiele levels are dependent on their previous experiences with subdomains of geometry. That is, a student's van Hiele level is not a global level for all of geometry.

Another important notion can be derived from comparing polygon and motions van Hiele levels. Overall, 65% of the students increased their motions level from pretest to posttest, 29% stayed the same, and only 6% declined. In contrast, 15% of the students increased their polygons level from pretest to posttest, 78% stayed the same, and 7% declined. Taken together with the observation that the students' motions levels were, for the most part, at or below their polygon levels on the pretest, this data suggests a hypothesis. That is, as students are introduced to new geometric material, although their initial reasoning about such material may be lower than their reasoning about geometric material they are more familiar with, their level of thinking about the new material quickly moves up to the level of thinking on the familiar material. This hypothesis is also supported by a detailed qualitative study that is described in a succeeding section of this chapter.

Finally, we return to a hypothesis we made earlier that students' study of motions at the end of the *Logo Geometry* curriculum may have caused them to obtain a lower polygon van Hiele level. For example, refer to the "Triad sorting" under "Interview" in the "Shapes and Levels of Geometric Thinking" section of this chapter. In addition, the triad choices and the reasons for the choices of one second-grade, case-study student (shown in Table 37) support this claim. This student's interim interview shows her focusing more on the components of figures, even grouping some pairs together using classification reasoning (although clearly not reasoning at Level 3). This progress is consistent with our observations of her in the case studies, and this increased attention given to components of figures continued into the posttest. Moreover, her responses also changed to reflect her use of visual transformation as the primary reason for grouping two figures together, but four of these responses dealt

[11] Control students did not take the pretest; posttest *LG* means (e.g., 1.16 vs. 1.14) are different because only *LG* students for which there were control students from matched classes were included in the *LG*/Control comparison analysis.

with transforming *components of figures* rather than whole figures. Indeed, her progression of responses for triad 4 show that she used geometric motions to try to make sense of the classification of a square as a rectangle. We suspect that the motions unit made reasoning based on visual transformation more salient in her thinking. (See the Coletta episode on "stretchy square bathroom things" in the final chapter.)

Motions Sorting Tasks. Two sorting tasks were used to assess the prominence of the slide, flip, and turn motion concepts in students. In the first, students were shown a stick-figure horse. They were then asked to sort cards depicting slide, flip, or turn images of the horse so that the cards in each group were alike in some way. Then they were asked to explain their sorting. Students were said to have a perfect slide sorting if they put all of the slide congruent images into one category; similar criteria were used for perfect turn sorting and perfect flip sorting. Students were said to have a perfect motion sorting if they had perfect slide, flip, and turn sorts. Only the slide category showed a significant difference between the *LG* and control groups ($\chi^2(1, 225) = 24.73$, $p < .001$), with 53% of *LG* and no control students having a perfect slide sort. The percentage of *LG* students that had perfect motion, slide, flip, and turn sorts on the posttest interview were: 2, 53, 3, 3 for 1988–89; and 0, 14, 6, 0 for 1987–88, respectively.

In the second sorting task, students were asked to sort the six tiles shown in Figure 27, which were presented in random arrangements. If they sorted perfectly by motion (e.g., into the columns shown in Figure 27, slides on the left, flips in the middle, turns on the right), this was recorded. If they sorted into other categories, such as by shape (e.g., top row versus bottom row in Figure 27), they were asked if they could think of another way to sort.

For 1988–89, 14% of the *LG* students versus 4% of the control students sorted the tiles perfectly into slide, flip, and turn groups on the posttest ($\chi 2(1, 225) = 6.89$, $p < .009$). Six percent of the *LG* students sorted the tiles perfectly into slide,

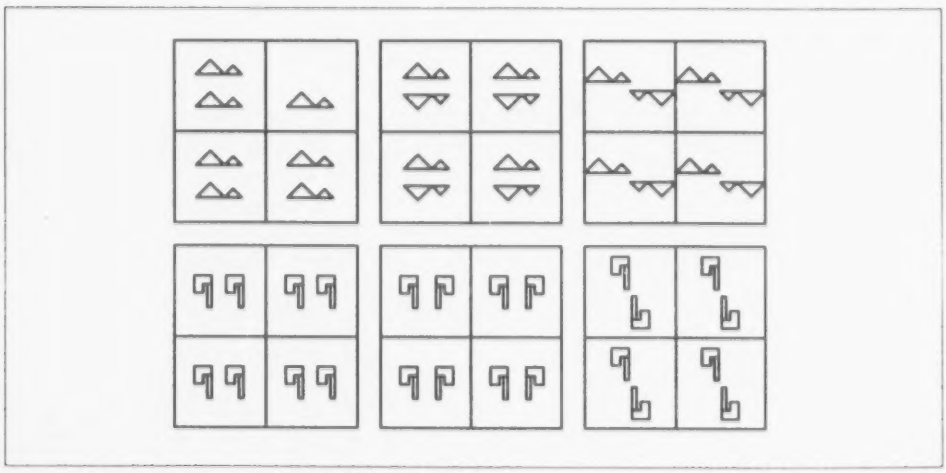


Figure 27.

flip, and turn groups on the posttest in 1987–88. For 1988–89, 18% of the *LG* and 9% of the control students sorted the slide congruent cards into one group ($\chi 2(1, 225) = 3.57$, $p < .06$); 15% of the *LG* and 4% of the control students sorted the flip congruent cards into one group ($\chi 2(1, 225) = 8.69$, $p < .004$); and 21% of the *LG* and 10% of the control students sorted the turn congruent cards as a group ($\chi 2(1, 225) = 5.23$, $p < .03$). The percents were 6%, 7%, and 12%, respectively, in 1987–88. When the students were asked to sort the cards a second time, the *LG* group had a slight increase of perfect sorts over the control group (but not significantly so). On average, the number of *LG* students making perfect sorts the second time was about 10% more than the number of *LG* students who did so on the first sort. Overall, the sorting tasks indicated that geometric motions were more salient for the *LG* students.

Classroom Observations

Across all grades, kindergarten through sixth, students showed every sign of understanding the slide, flip, and turn concepts, terminology, and computer commands within the context of the curriculum. They became proficient at identifying which motion was called for to move one figure onto the other; further, they seemed to comprehend the relationship between these motions and the determination of congruence. They even were able to see the inverses of these motions without prior instruction (recall that the curriculum emphasized "undoing," as when students reversed the commands to bring the turtle back along a path).

For example, students' competence was displayed when a curriculum coordinator for one school district, who was previously a mathematics teacher, observed a second-grade class. He observed a problem two students were contemplating, and suggested that they would have to turn the object and then slide it. The students looked at him in puzzlement and one remarked, "But one flip would do it and that would be faster!" He beat a hasty retreat and thereupon mentioned to the project directors that these second graders were more competent in motion geometry than a class of eighth graders had been after he had taught them a unit on the topic several years before.

PROBLEM SOLVING AND SENSE MAKING: CLASSROOM OBSERVATIONS

The main goals of *LG* are to develop not only students' geometric sense making but also their mathematical problem-solving skills. To promote both goals, instructional tasks were presented as problem-solving activities, many of which arose from careful observation and analysis of students' work during the field testing of the curriculum. That is, during our field testing, we found that ostensibly simple tasks often presented rich and challenging problem-solving opportunities to students. In this section, we will describe several tasks and episodes that illustrate the problem-

solving and sense-making nature of the *LG* curriculum. In these episodes, taken from our in-depth case studies, one or the other of us (Battista or Clements) was in the classroom every day as the curriculum was taught (and is denoted as the "Observer" in the dialogue). We were not there to teach, only to observe. Our questions were aimed at probing students' thinking.

"Tilted" Squares

We will consider first the seemingly simple task of getting the Logo turtle to draw a shape in a nonstandard orientation. In one second-grade session, only two of the seven students reasoned that the procedure they had developed for drawing a square with vertical and horizontal sides could, with the addition of a single turn command, be made to draw a "tilted" square. The other five students completed the tilted square visually—choosing a turn, trying it, visually checking to see if it was correct, and revising their commands if necessary, often doing this for each of the square's vertices.

As we observed this and similar episodes, we arrived at two important conclusions. First, many students had abstracted far less about the properties of squares than we had first assumed—they clearly had not abstracted that squares have four 90° angles. Second, making a tilted square after having made a standard square was a perfect problem-solving task for these students at this time. Indeed, clearly these students still had far to go in their conceptualization of the properties of squares, so working on this problem not only involved the students in problem solving, it encouraged them to make progress in their conceptualization of squares. Moreover, the task of making a tilted square was intrinsically interesting to students—some even posed it for themselves. And, as we observed students working on this task, we judged that the conceptualizations needed to solve this task were, with appropriate time and guidance, within the students' current capabilities. In fact, several weeks after the initial episode, two of the students who initially approached the problem visually said that the turns in a tilted square would be 90.

Turtle Deliveries

Student Task: Given the on-screen map shown in Figure 28, write a procedure to teach the turtle to deliver a load of pies to the restaurant R1 and to return home. The path home from the restaurant must be the same as the path **to** the restaurant. Do the same for restaurants R2 and R3.

Students were working in a large group with the teacher who was using an overhead monitor to display the computer screen. As a group, the students had given commands to move the turtle to R1. These commands were entered into the Logo editor as follows:

```
TO GO.R1
LT 130
FD 40
```

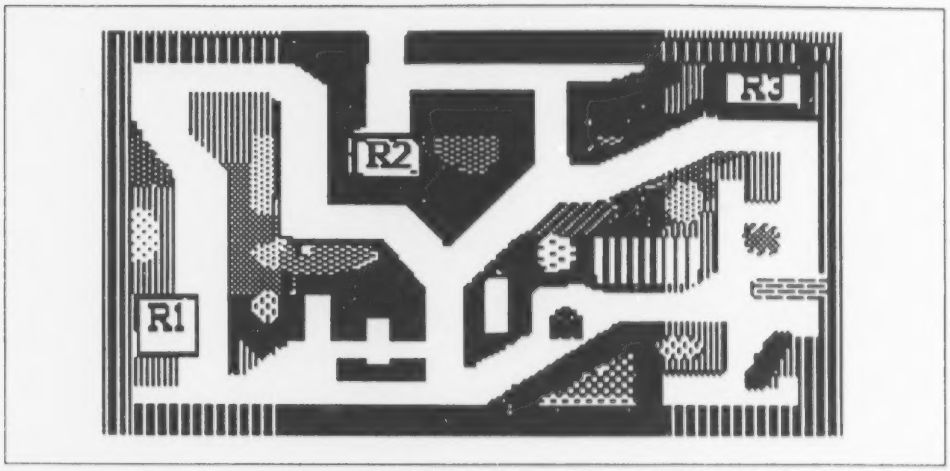


Figure 28.

RT 25
FD 75
RT 90
FD 30
LT 90
FD 10
END

This set of commands was run so that all the students could see the result. Because their path to R1 was incorrect, the students discussed what Logo commands should be changed. The revised command set is shown below, with changes in bold type.

TO GO.R1
LT 130
FD 40
RT 40
FD 85
RT 90
FD 30
LT 90
FD 10
END

After running the revised procedure, the students were satisfied that it was correct. The next part of the task was to get the turtle to return to its starting position along the same path.

Teacher: How are we going to get the turtle back?

Andy: Everything that was a forward is a backward, and everything that is a right is a left, and everything that is a left is a right, probably.

Teacher: So, you're telling me what?

Steven: Reverse the stuff.

Andy: You reverse, go from the bottom [of the procedure] to the top.

Teacher: What comes first?

Steven: BK 10, RT 90, ...

[So the students are correctly giving the commands to undo the path.]

Andy: We've got a problem here, because here's the restaurant. It [the turtle] goes out of the restaurant [using] BK 10. Then [going] right would be up here, so it's going to get screwed up.

Another student: Now it's going **backwards**.

The students discuss Andy's misgiving about the proposed solution. Andy figures out his mistake as he stands up and acts out the commands; he sees that you need a right turn because, in the next command, the turtle is going backwards (he envisioned the turtle going forward). The commands are then entered into the computer, and the students see that their proposed solution is correct.

Throughout this episode, the teacher entered the students' commands into the computer and moderated the discussion. She did not judge the correctness of the students' ideas; instead, the students judged correctness by examining the results on the computer screen. Judging the validity of one's proposed solutions (as opposed to asking the teacher if the answer is correct) is an essential component of a genuine problem-solving activity.

The teacher then sends the students to the computers to continue the activity in pairs. Jill and Peter, working together at a single computer, immediately attempt to get the turtle to the second restaurant, R2. They use the turtle turner and ruler to make measurements on their activity sheet and enter the commands into the editor. When they run their procedure, they find that it does not get the turtle to R2. Jill tries to fix their bugged procedure by analyzing the path it makes and conjecturing about where their path is erred, ("We need to go 20 more right there.") Peter, in contrast, wants to give commands to the turtle in "immediate mode" so the effects of each command can be seen as it is entered. Apparently, he has decided that their original procedure has so many mistakes that starting over will be more efficient. Jill accedes to Peter's strategy, so the students give successive commands to the turtle in immediate mode, recording their commands on paper.

The next day, as students continue their work on the path for Restaurant 2, Jill and Peter enter their recorded commands into the Logo editor, run the resulting procedure, and observe that the path is not quite correct. Interestingly, Jill and Peter express no surprise or disappointment when their procedure does not work.

They anticipated that their path would not be perfect and that they would have to fix it. Jill and Peter then identify the part of the *path* that needs to be fixed, carefully locate the corresponding *command* in the path's procedure, then change the command. When they try the new path, they are satisfied that the corrected path works.

In this episode, Jill and Peter exhibit a number of processes and characteristics important for successful mathematical problem solving. They show confidence in their ability to solve the problem, persistence in the face of obstacles, expectations that initial attempts at solving a problem might fail but that such attempts can be productively altered (the majority of students in a standard curriculum simply give up if their initial attempt at a solution is incorrect), and use of metacognitive skills in deciding what approach to take in fixing the error. They effectively use the important problem-solving sequence of trying something, reflecting on how it works, then making appropriate revisions. Finally, because they are in an instructional environment that encourages it, Jill and Peter exhibit intellectual autonomy in solving this problem.

Immediately upon seeing that their solution is correct, Jill says, "Let's get him back." Both students are excited about tackling this new problem; it is not a typical school task that they attempt with little enthusiasm. As they start on the problem, initially, Jill thinks that the first command in getting the turtle back to its starting point is the reverse of the first command in the procedure for getting the turtle to R2. But Peter says that it must be the reverse of the last command.

Observer: Why does it have to be this [the last] command? [Simultaneously, Jill and Peter offer several explanations, all of which seem to have correct elements but are incoherent. Finally, Jill explains.]

Jill: We started from here [turtle's starting point] and ended here [draws in the air a path that is not straight but not a copy of the screen path either (almost a curved version)]. So we want to start here this time [indicates R2] so we have to start with the last one [points to last command in original procedure].

Even though the idea of undoing a path had been discussed the previous day, Jill and Peter needed to try it themselves to make the idea personally meaningful. After they had tried it and thought about it for themselves, both seemed to make sense of the previous day's explanation of reversal and were able to apply it. Jill's explanation indicates very clearly that she possessed an abstract conceptualization of the process of undoing a path. She was able to indicate, by drawing in the air, that she possessed a visual abstract image of a path and could operate on that image; she did not need to go back to the particular path that they were drawing. Jill was clearly operating on an abstraction of a path, not only because she *imagined* operations on the path, but also because she loosely drew her "air" path without trying to preserve the exact shape of the screen path. (A fifth grader in a different class also

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evidenced a clear understanding of the process of undoing. He explained it to his partner as follows: "Like if someone was going to go down the road, and they pulled out of your driveway to get somewhere and they end up at a dead end, they wouldn't just go forward, even though the first thing they did was go back out of your driveway.")

This episode involving Jill and Peter illustrates the synergistic roles that students' individual and large-group work played in their geometric knowledge construction in *LG*. Evidently, class discussions often offer students incomplete blueprints for knowledge construction; the actual building of the knowledge structures does not occur until individual students have sufficient opportunities to engage in personal sense making. *LG* encourages and supports student sense making as students personally manipulate ideas and decide on their validity and exact form. This episode also illustrates the opportunity that *LG* provides for students to form and operate on mathematical abstractions.

> Jill and Peter now enter their commands for a procedure to get the turtle from R2 back to the starting point. They try it, find it incorrect, but do not seem to be able to locate the error. However, even though they are stuck on the problem of fixing their procedure, they persist in their problem-solving efforts. Finally, they carefully check their procedure in the editor to see if they have properly reversed all the commands. They find and correct a mistake in a turn command, try the revised procedure, and are excited to find that it works.

As a final challenge in "Turtle Deliveries," students are given a procedure which takes the turtle to OILERS, an "out of town" restaurant; that is, to a location off the screen and out of sight. The challenge: Bring it back.

Jonathan:	How can you get back from OILERS? Is there a map?
Kevin:	No.
Mary:	You just reverse them!
Jonathan:	Oh, yeah! Without even seeing.
Kevin:	That's weird! It really will work!

Kevin's last comment indicated his amazement that a logically-derived solution, divorced from visual affirmation, can accomplish a task. This task, in addition to encouraging students to raise their initial "Turtle Deliveries" conceptualizations to a higher level of abstraction, also helps students appreciate the power of abstract mathematical models of phenomena.

In sum, what sense-making and problem-solving processes were students involved with in the "Turtle Deliveries" episodes? First, the "Turtle Deliveries" activity represented a significant challenge for students; it is not a task that can be completed in 3 or 4 minutes, as are most school mathematics tasks. Students had to develop sufficiently robust mental models of the situations to contemplate the required turtle paths. They had to apply their concepts of lengths and turns. They had to maintain the proper correspondence between symbols (turtle commands) and

geometric entities. They had to test possible problem solutions, conjecture about mistakes, and decide on the best strategy for proceeding (an important metacognitive activity). Moreover, both the computer environment and the classroom culture promoted and supported intellectual autonomy in students' mathematical sense making and problem solving. This is in sharp contrast to traditional instruction (Stigler & Hiebert, 1999), in which students simply attempt to determine what the teacher would like them to say (i.e., using the teacher as the source of mathematical knowledge rather than their own thought).

Making Equilateral Triangles and Regular Polygons

Student Task: Complete the procedure below so that it draws an equilateral triangle.

```
TO EQTRI
FD 50
RT __
FD 50
RT __
FD 50
RT __
END
```

> At their computer, Jill and Peter first try 90 for the turns, then 45. They conclude that 45 is too *small*. Peter says that the turtle has to turn first to make an equilateral triangle. When Observer asks what happened when they tried 45, they say it is too *big* (note that originally they thought it was too small.) Jill says, "If that's 45 (she motions with her hand turning from the vertical) and that's 90 (motions), then it would have to be down more (motions about 135)." (Note how Jill is visualizing the turns. She clearly understands the turning motion and has become quite adept at estimating turn measure.) After they try 135, Peter says the turtle needs to turn more; Jill less. Jill says she is confused. Peter then wants to turn the triangle, so he inserts an initial turn command into their procedure. Jill says that an initial turn is unnecessary, "Because it's a straight line and you turn, it's not going to make any difference." Peter then claims that the procedure needs first a 45-degree turn (to change the orientation), then 145-degree turns after that (to form the vertices). Jill counters that all the turns have to be the same.

At this point in the problem-solving activity, Jill and Peter are dealing with a number of conceptual conflicts. Peter thinks that the equilateral triangle cannot have a vertical first side. Jill correctly believes that the first turn is irrelevant. However, Jill gets confused when Peter makes the first turn 45 and subsequent turns 145—she is thinking about the requirement that all the turns in an equilateral triangle are

the same. But she has not made complete sense of the requirement; her understanding of it is easily unraveled. Jill is also trying to make sense of Peter's ideas—the students are productively interacting. Finally, both Jill and Peter are having difficulty deciding whether an amount of turn is too much or too little.

> When Jill and Peter try 145, they conclude that it is too much of a turn. Also, as a result of seeing their procedure drawn on the screen, Peter changes his mind about the relevance of the initial turn, saying, "It might just be upside down." [Because Logo allows students to test the validity of their ideas, Peter finally has recognized his mistakes.] Jill and Peter then try 115 and think that it works. But when the observer asks them to hide the turtle, Peter thinks 115 is too much. After some further experimentation in which they discover the relationship between the turn input and the size of the angle the turn creates in the path, Jill and Peter finally arrive at 120° turns, exclaiming simultaneously, "Yeah."

Observer: So, what was the total amount of turning?

Peter: 360 degrees. Ohhh [very excitedly]. We should have just taken [Jill shrieks, too]; we should have divided 3 by 360 [meaning 360 by 3], and we would have gotten 120.

Observer: How come?

Peter: You have to turn 120 three times to get 360.

Observer: What's so special about 360?

Peter: It's a full circle. Ohhh [groans].

Jill: We were so stupid.

These two students were truly excited about their discovery. But notice that they did not come upon this discovery by logical deduction, nor is it certain that the observer's question, which is on the students' activity sheet, would have evoked this same flash of insight before they had played with the ideas numerically. They needed to manipulate the ideas in their own way, to work out their conceptual conflicts.

Also note that Jill and Peter's problem-solving efforts were initially hampered by their conceptual confusion about whether the amounts they turned the turtle were too much or too little. It was not until they developed an appropriate conceptualization of this geometric concept that they could succeed in their problem-solving efforts. Thus, this activity not only involved students in significant problem solving, but it also helped the students correct and sharpen this important geometric concept. Indeed, it is highly likely that this experience helped Jill and Peter better understand the difficult (for this age student) relationship between the two concepts of turn along a path and the angle formed by the path.

The confusion initially experienced by Jill and Peter might be disturbing to many teachers. But it is actually an important part of mathematical learning, sense making, and problem solving. Jill and Peter struggled to make sense of what for them were very difficult ideas. It was only through their experimentation with the computer and their discussions with each other that they are able to make sense of

the ideas for themselves. Thus, overall, this episode is a good illustration of the constructive and problem-solving nature of the *LG* lessons, of how mathematical inquiry and discovery can be extremely exciting for students, and of how students can find and correct their own mistakes in the proper environment.

Generalization of the Angle Sum Notion

Even though Jill and Peter made an important discovery—that the turn for an equilateral triangle is 360/3—we must still question whether they sufficiently abstracted and generalized the idea so that it can be applied in other problems. We get a chance to investigate this question two days later as the class is attempting to write procedures for other regular polygons.

The students are at the computers in pairs. They are to write a procedure, first, for a square, then for a regular pentagon, and so on, using the REPEAT command. Jill and Peter start with the regular pentagon.

Peter: We gotta figure how much the rights have to be.... The square is 90. [They use a turtle turner to measure the turns for a regular pentagon on their activity sheet.] It's always the same angle. [Jill measures and gets 55. So, they try RT 55 in their procedure, but it doesn't work.] Well.

Jill and Peter use their turtle turners to measure, respectively, the screen and the activity sheet pictures of regular pentagons. [Note again how Jill and Peter persist in their problem-solving endeavors. Peter's response ("Well") indicates that he is puzzled. But they proceed undaunted.]

Observer: What was wrong with 55?

Jill: I did it like this and it was at a slant. [She did not turn the turtle turner to the correct initial heading.]

Peter: [After first saying it's about 63] Wait, I messed up. Almost 90.... 85. [After trying this turn on the computer and viewing the result] That's too little. [It was really too much.]

Observer: Is that too much or too little?

Jill: Because, if it was 90, it would make a square [tracing the figure on the screen with her finger]. I think it's too much.

Jill's reasoning was quite sophisticated. If 85 is too little, then 90 would be moving in the correct direction. But 90 would give a square. So, 90, and therefore 85, is too much. (Note that, as with the equilateral triangle, Jill and Peter are still having difficulty deciding whether a given amount of turn is too much or too little.)

Jill: Let's try 80, do you want to?

Peter: Yeah. Maybe we need to, the turns are all wrong. [Peter mumbles in the background—he seems puzzled.] If this doesn't work, it's probably been too small.

Jill: [They try 80.] It's closer.

Peter: Yeah.

Jill: So, it would be too much.

Peter: [As they go back to the editor] Wait. Divide 5 by 360. You'll get the right answer. [Peter means divide 360 by 5, a common mistatement for this age level student.] [After they do the division] Same as we forgot to do last time [when they did equilateral triangles]. [After computing the quotient of 72, entering it into their procedure, and running the procedure.] It worked.

Jill and Peter immediately go on to the hexagon problem.

Jill: Six divided by 360. [Of course she means 360/6.]

Peter: Sixty. [Pause.] Oh, so, it goes down every time.

Jill: Here we go.

Peter: It's going to work. [When they run their procedure for the regular hexagon, it works correctly].

At this time, the classroom teacher, who has been circulating around the classroom, interacting with other pairs of students, arrives to check on Jill and Peter.

Teacher: What did you figure out?

Peter: You have to divide the number of sides by 360, I mean into 360. [Jill was explaining almost simultaneously.]

Teacher: Why?

Peter: Because there's going to be three angles, and you need to turn 360 degrees, and that will tell you how much to turn.

Jill: Not three angles.

Peter: Well, I mean how many angles are in the figure?

Teacher: Why 360?

Peter: Because that's how many degrees it takes to go around a circle. [As he says this, Jill moves her finger in a complete circle on the screen.]

Jill and Peter now go back into the editor and write a procedure for a regular octagon. They calculate that 45 will be the correct turn. It works, but seeing it drawn on the screen seems anticlimactic to them.

Even though Jill and Peter had previously discovered that 360 should be divided by the number of turns for equilateral triangles, they did not generalize this idea beyond that specific situation. They needed, and were provided with, opportunities for further investigation—a chance to further abstract and generalize their earlier discovery. Also, their first approach to the problem was again to play with the numbers (a good strategy when facing a new problem that we certainly promote in the curriculum). However, Peter seemed to be bothered at one point because he thought they should be able to figure it out rather than use the guess-and-test technique. That is why he was mumbling to himself.

In addition, Jill and Peter were certainly reflecting on their activity, as indicated, for instance, by Peter's comment, "Oh, so, it goes down every time." More important, Jill and Peter seemed to be moving to a higher level of reasoning. They had not only discovered that the rule of 360 could be used to figure out the amount of turn, but they were also convinced that their method made logical sense. For instance, testing their answer for a regular octagon seemed perfunctory to them;

the logical structure had convinced them that their answer had to work. This is a major accomplishment for students of this age. In fact, even most high school students are "naive empiricists" and believe that anomalies can occur even after a theorem has been proved.

Later the teacher has all the students report and discuss their individual findings at the computer.

Teacher: What do you think the turn will be for a decagon?

Many students: 36

Teacher: Tell us, Peter, about your discovery.

Peter: [Hesitatingly] You have to divide the number of sides of that figure by 360 and you'll get the number to turn.

Peter was still not very articulate about his discovery, but he clearly understood it. Indeed, it was common for students to discover ideas that they had difficulty putting into words. Giving students the opportunity to verbalize their ideas in classroom discussions is an excellent way for them to clarify and consolidate their thinking and is also consistent with NCTM's recommendations to support students' ability to communicate mathematics.

Teacher: Why do you think Peter got 360?

Andy: It's like a whole turn.

Teacher: What do you mean a "whole turn"?

Andy: Like the turtle always ends up in the same place. So, divide how many turns into where it stands up.

Teacher: Why do you think that works with the turtle making a whole turn?

Cary: There's 360 degrees in a circle.

Several days later, Jill and Peter are doing the "Sum of the Vertex Angles in a Polygon" activity, as is described below.

Student Task: Each procedure draws a polygon. For every polygon, figure out and record the measures of all of its vertex angles. You can check you answer for the first triangle by typing T1, turning the amount that you answered, then giving the FD 60 command. If your answer was correct, the turtle should be at the top vertex of the triangle. Try to discover a relationship between the number of sides of a polygon and the sum of its vertex angles. **HINT:** Think about the Total Turtle Turn Theorem.

POLYGON	NUMBER OF SIDES	MEASURES OF VERTEX ANGLES	SUM OF VERTEX ANGLES
EXAMPLE:			
TO T1	3		
FD 60			
RT 90		90°	
FD 80			
RT 143		37°	

```
FD 100
RT 127 ..... 53°
END
    Total Turtle Turning 360°
TO T2                    _____
FD 71
LT 135 ..... _____
FD 50
LT 90 ..... _____
FD 50
LT ___ ..... _____
END
    Total Turtle Turning _____
```

> As Jill and Peter start to measure turns, the observer asks them what their plan is. The students explain that they are going to measure the turns to find out what to use in their procedure. Then Jill says to the observer, "Is there another way?"—to which he replies that they can figure it out without using their turtle turners. Jill then says, "Use 360."

Thus, using logical reasoning, as opposed to empirical methods, was well within Jill's grasp. However, as in the previous activities, there was a natural inclination for students to start with an empirical approach. Jill would not have thought of trying to use logical reasoning initially without the observer's intervention; it was as if his question about the solution strategy was a metacognitive prompt for her to think about and evaluate possible strategies before implementing them. Schoenfeld (1992) talks about similar needs for students in college! By regularly repeating prompts such as these, teachers can help students develop important metacognitive problem-solving skills. But it should be emphasized that such prompts will not be useful to students if they are not given the opportunity operate at their own level. That is, students must be cognitively "ready" to make personal sense of such prompts.

Chapter 4

Discussion

The Logo Geometry Project attempted to ameliorate problems in geometry teaching and learning by developing a research-based Grades K–6 geometry curriculum. This chapter discusses the results of evaluating this curriculum with 1,624 students and their teachers and, more widely, discusses the results of investigating how elementary school students learn geometric concepts and what role Logo programming in turtle graphics played in students' learning of geometry and spatial sense.

TOTAL ACHIEVEMENT

Across Grades K–6, *Logo Geometry* students scored significantly higher than control students, making about double the gains of the control groups on the pre-post paper-and-pencil assessment (e.g., the K–6 Total Scores). These findings are significant for two reasons. First, in designing these assessments, we took the conservative stance of testing in a paper-and-pencil mode—the format in which the control students did their geometry work. That is, *LG* students had to transfer any knowledge developed in (and thus potentially linked to) the computer context to paper-and-pencil items. We also avoided paper-and-pencil items connected directly to the *LG* curriculum. Second, the curriculum is a relatively short intervention, lasting only six weeks. We expected that students who received rich geometry instruction for a greater part of the year, and more significantly, for several consecutive years, would perform even better.

Examination of analysis of variance interactions did not provide convincing evidence that students whose teachers had two years of experience teaching the *LG* curriculum performed differently on paper-and-pencil tests than students whose teacher had one year of experience.

Similarly, *LG* students in Grades 4–6 outperformed control students on the composite score, which included all items administered to students in these grade levels (Table 4, 4–6 Total Scores). A conjecture emerges in comparing these two composite scores. Recall that the K–6 Total composite score included only items that were administered to all students, whereas the 4–6 Total score also included these items plus the more difficult items only administered to students at higher grade levels. Although *LG* students in Grades 4–6 appear to perform moderately better on the former composite, they outperform the control group at every grade and site on the latter (see Table 3). In the extreme example, Site 2 Grade 4 *LG* students did not make significant gains and performed significantly lower than the Site 1 control group on the former score but made significant gains and outper-

formed both control groups on the latter score. Thus, intermediate-grade students may have benefitted more from the *LG* curriculum in areas tapped by the more demanding, higher level items included only on the 4–6 Total test.

SHAPES AND LEVELS OF GEOMETRIC THINKING

Three items, adopted from previous van Hiele-based research, assessed the identification of triangles, rectangles, and squares within a large set of distractors. *LG* students performed better overall, with gains attributable to the triangle and rectangle items (of all the groups, only the kindergarten *LG* group made significant gains on the square items).

The particular items on which *LG* students performed better than the controls were investigated. *LG* students, especially younger students, gained more than controls—not in their ability to identify figures that are triangles—in their ability to correctly identify nontriangles that share visual characteristics with triangles (e.g., a chevron). This was especially salient for the concave quadrilateral and for figures that appeared to be triangles but had curved sides or were not closed. Patterns of responses for these figures show a movement from unsophisticated, holistic visual responses to responses that were based on a more accurate consideration of the components and properties of the figures. *LG* may enhance students' triangle scheme with imagistic constraints, based on turtle movements with appropriate movement-based mental models. For example, imagistic constraints for triangles may include three forward movements that create a sequence of three connected straight paths that form a figure that ends where it starts (and thus is closed). Such constraints may prompt corresponding reflections on the components and properties of triangles; for example, that three line segments form a closed figure. The lack of differences between the groups for identification of triangles, even those in nonstandard orientations, argues that the positive effects of *LG* may not lie in the development of basic visual prototypes for triangles. Because differences were noted on the ability to correctly identify nontriangles that shared visual characteristics with triangles, the positive effects of *LG* were more probably in the enhancement of conceptual knowledge. That is, as in Reif's (1987) model of interpreting concepts, visual prototypes are used, but in unfamiliar situations or when possible inconsistencies arise, they are checked with more formal, rule-based, knowledge. This latter type of rule-based, conceptual knowledge may be what students in *LG* classrooms developed.

This is consistent with other effects. For example, the smaller effects for squares would be expected in that the class of triangles has members that are less easily assimilated to a small number of visual prototypes than does the class of squares.

For rectangles, large growth was displayed for *LG* students, compared to control students, for the two squares. For both of these shapes, the most striking growth occurred in Grades 4, 5, and 6. This corresponds to the *LG* instruction at those grade levels, which attempted to help students understand (via properties) the hierarchical classification concept of squares as special cases of rectangles. Similarly, moderate

effects on nonrectangular parallelograms, especially those in nonstandard orientation, indicated that *LG* instruction encouraged children to evaluate shapes by properties rather than by visual appearances.

Unit tests for *LG* students (Grades K–4) confirmed results for the triangle, rectangle, and square items and suggested that only at Grade 4 did students begin identifying squares as rectangles spontaneously. Further, only students in Grades 5 and 6 performed well on items involving regular and nonregular polygons, although this may have been due to a lack of emphasis on naming and working with such figures in the lower grades.

LG students improved more than control students on attributing statements of geometric properties to the classes of squares and rectangles. For the rectangle concept, effects tended to be strongest for students in Grades 3–5 and weakest for Grade 2. It may be that students younger than Grade 3 have more difficulty abstracting certain properties from their work with *LG*. Over the grades, *LG* had the strongest effect, compared to the control instruction, on the properties dealing with paths and with angle measure. *LG* students outperformed control students on the property "opposite sides equal"; however, this difference was large for squares but small for rectangles. Thus, the differences between the groups do not relate to familiarity with the property, per se, but to understanding this property's consistency with the property of "all sides equal in length." *LG* students could apply both properties to the class of squares. This ability demonstrates flexible consideration of multiple properties that may lay groundwork for later hierarchical classification.

By synthesizing the results with regard to hierarchical classification, we conclude that important precursors for hierarchical classification are developed by *LG* instruction. Still, students have difficulty integrating these preliminary ideas into a property-based hierarchical classification system. In other words, most students are learning to think about these shapes at Level 2 but are not sufficiently knowledgeable about the properties of these shapes to construct viable Level 3 systems of relationships. Why, then, did the first rectangle item seem to indicate that *LG* instruction helped substantial numbers of students identify figures in a manner consistent with the hierarchical relationship? When *LG* students succeeded in identifying squares as rectangles on the rectangles item, they could have done so by asking themselves if a "Rectangle" procedure could have drawn each of the given shapes. These students may have been able to use their experience with Logo procedures to overcome previously held exclusive conceptualizations of these figures while still not being able to deduce relationships from more formally stated properties.

Observation revealed that the Logo microworlds proved to be evocative in generating thinking about squares and rectangles for young students. Some kindergartners formed their own concept in response to their work with the microworlds. For example, the "square rectangle" phrase, which was generated by several students besides Chris, was appropriated by the teacher of that class. This concept was applied only in certain situations. Squares were still squares, and rectangles, rectangles, unless you formed a square while working with procedures—on the computer or in drawing—that were designed to produce rectangles. The concept

was strongly visual in nature, and no logical classification, per se, such as class-inclusion processes, should be inferred. The creation, application, and discussion of the concept, however, was arguably a valuable intellectual exercise. Even for students such as Robbie, who rejected such naming, the disequilibrium that the microworld engendered was a valuable stimulus for reflection.

Observations of fifth graders (Jonathan and Jeremy) are consistent with the conjecture that students held multiple conceptions, some of which were evoked and applied only in certain situations. Jonathan's reasoning sometimes was similar to that of kindergartners who thought that squares and rectangles usually were separate unless the former was produced by a procedure designed to draw the latter. At other times, Jonathan would relate these classes of shapes, but only when prompted by a teacher to use property-based reasoning. This indicates that Jonathan had not achieved Level 3. However, the Logo environment helped him and other classmates generate alternative conceptions that may conflict with ways of thinking that were previously established. These alternative conceptions may eventually serve as a catalyst for the development of higher levels of geometric thinking. The creation of such concepts supports the hypothesis that certain Logo experiences may lead to situation-specific concepts and behaviors that are consistent with hierarchical classification, without being built on property-based relationships as described by van Hiele. In contrast, fifth-grader Jeremy constructed ideas much closer to established mathematical relationships. He gave justifications that were based on the properties of squares and rectangles. Further, he applied similar reasoning to other classes of quadrilaterals, including those whose hierarchical interrelationships were not studied by the class. In sum, it appears that *LG* can help students from kindergarten to Grade 5 visualize, reflect on, and discuss geometric figures and relationships. Although most students will begin to learn about properties, even achieving strong Level 2 thinking, others may even construct hierarchical relationships that indicate Level 3 geometric thinking. Slow movement to such thinking is reasonable, considering that partitioning shapes into separate classes is not only the societal norm but is as logically consistent as hierarchical classification (de Villiers, 1994). Thus, resistance to hierarchical classification does not necessarily align with the logic of the inclusion but often aligns with the meaning of the activity, both linguistic and functional—*linguistic* in the sense of correctly interpreting the language used for class inclusions and *functional* in the sense of understanding why it is more useful than a partition classification (de Villiers, 1987, 1994).

The triad task of the interview, which asked students to choose two of three shapes that were most alike, was also designed to elicit students' level of geometric thinking. Across the nine items, the pattern was for *LG* students to move from visual to property or classificatory responses. However, the predominance of visually based reasoning at all administrations implies that this movement was in the beginning stages. The movement is consistent with what we predicted in our theoretical analysis. For several problems, there was also an apparent shift from giving (visual) components as reasons to using visual/spatial transformations to justify answers (cf. Lehrer, Jenkins, & Osana, 1998). As we argued earlier, the motions unit,

which students studied between the interim and posttest interview, made students more aware of how one shape could be moved or visually changed to look like another. Although students' choices of figures changed little from the pretest to posttest, the *reasons* for those choices changed, sometimes dramatically. And as was expected, these shifts in level of thinking occurred in older students more often than in younger students.

Even though noteworthy progress was made on many of the triad items, there were other items for which visually salient similarities attracted the majority of students. The advantage of viewing figures in terms of properties or classifications should be made more evident to students to encourage them to abandon their visual focus. Indeed, perhaps in many mathematical situations, students did think of the figure in terms of properties. However, they probably saw no larger advantage to employing a logical-mathematical approach to the triad task. So, they were easily swayed to choose the pairs that matched best visually. Finally, it must be kept in mind that our triad classification system, because it emphasizes properties and classification of static figures, is biased against visual transformations as evidence of sophisticated geometric thought. However, thinking about figures in terms of such transformations is sometimes at the height of geometric insight. *In summary,* it is important to note that, using a first draft of the *LG* curriculum, the triad data suggest that there is a progression in levels of thinking, that takes considerable time, as measured by a far transfer task.

Also, our findings indicate that students improved on a task that required complex reasoning about the definitions of quadrilaterals. However, as measured by the triad task, students' performance decreased from interim to the posttest, consistent with the decrease in van Hiele levels. Thus, students made progress moving to more formal thinking about shapes during the shapes module but regressed slightly afterwards. More than likely, these findings are dependent on the van Hiele level of the students. Students in Grades 4–6 were generally in transition from Level 1 (visual) to Level 2 (properties). Therefore, they were likely to abandon the visual condition of "two long sides and two short sides" for rectangles and require instead that the figures have right angles. Still, students had difficulty with the square/rectangle classification problem, and they grappled with using the type of logical deduction required to deal with the various aspects of this complex item effectively.

Classroom observations indicate that, within the context of the *LG* curriculum, students learned to analyze the visual aspects of these figures and how their components are put together, thus facilitating a transition from the visual to the descriptive-analytic level of geometric thinking. The Logo environment and tasks provided students with opportunities to analyze and reflect on the properties of two-dimensional shapes. Also, the need to build Logo procedures allowed students to develop understandable, implicit definitions of these shapes. Further, the computer environment provided students affirmation that the conclusions that they reached through logical reasoning were indeed correct, which helped them build confidence in this nascent mode of thought. Thus, the computer experimentation afforded students the opportunity to gain confidence. In some cases, such as Jonathan's

budding Level 2 awareness, we see both of these advantages at work. His initial explorations with Logo and his "running through the procedure definition in his head" contributed to his emerging sense of the properties of shapes and of his certainty that a given shape was not a member of the class of rectangles.

Overall, results provide support for the hypothesis that *LG* is effective in developing schemes for basic geometric figures that improve performance on van Hiele-based assessments. This is consistent with other studies indicating that the appropriate use of Logo helps students begin to make the transition from Levels 0 and 1 to Level 2 of geometric thought (Butler & Close, 1989; Clements, 1987; Clements & Battista, 1992b; Hughes & Macleod, 1986). Also, as predicted, Logo experience encourages students to view and describe geometric objects in terms of the actions or procedures used to construct them (Clements & Battista, 1989). Consistent with van Hiele's predictions, however, most students did not move from one level to the next in a few weeks of instruction.

The ability to write a computer program to draw shapes also increased, though even relatively simple tasks remained difficult for many students. Thus, learning Logo programming had a facilitative effect on students' geometric conceptualizations. However, several caveats are also in order. For example, some degree of learning for some students may have been independent of their learning of turtle geometry. In addition, it might be that the *LG* curriculum was too short (recall that no class experienced the approach for more than several weeks of a single school year) to support complete learning of the relevant turtle geometry, and thus its effects on geometry learning were foreshortened. That is, as others from a spectrum of theoretical perspectives have stated for the case of intellectual skills, one should not expect large transfer without mastery of the initial domain from which transfer is predicted (Anderson et al., 1993; Clements & Merriman, 1988; Klahr & Carver, 1988; Kurland, Pea, Clement, & Mawby, 1986). Nevertheless, classroom observations reveal that the environments offered students opportunities to analyze and reflect on the properties of geometric figures. The act of writing Logo code to draw shapes and then confronting the conflicts between the relationships embodied within the Logo procedures and students' intuitions about figures was particularly influential in students' learning.

Two types of shape problems, disembedding (Grades K–1) and the building question (Grades K–3), were different in nature. The first, disembedding, was not a goal of *LG*. Results indicated that although kindergartners could accurately and comprehensively identify the basic shapes of square, rectangle, and triangle embedded in pictures, only a little more than a third of the second graders could do so when these shapes were embedded in complex geometric figures. As expected, overall, results indicated no beneficial effect of *LG* on this ability, but no evidence of benefit of the traditional curriculum occurred either.

Second, work with *LG* significantly improved primary-grade students' recognition of the relevance of arithmetic processes in the solution of geometric problems, and it improved their ability to apply these processes accurately (e.g., in the the K–3 Building question). Such gains, however, may more likely be observed

when students are comfortable with the numbers (measures) involved. *LG* students in Grades K–1 made greater gains than control students using single-digit numbers. Conversely, only Grade 3 *LG* students made substantial gains compared to control students for two-digit numbers. The notion that they must be "comfortable with the numbers involved" is conjectural and should be evaluated in future research.

ANGLES, ANGLE MEASURE, AND TURNS

Considering the critical role that turtle rotations play in forming geometric figures, it was expected that *LG* would facilitate the development of the geometric concepts of angle and angle size. *LG* students did not identify angles from a set of figures better than control students did. (Unit tests indicated that *LG* students could identify the number of turns and angles in geometric figures accurately; above 80%.) However, *LG* students did significantly outperform control students in drawing an angle, even though this difference was slight. The youngest and, to a lesser extent, the oldest *LG* students improved the most relative to control students. On the interviews, *LG* students drew more correct angles than the control group. The latter tended to draw shapes and tilted lines. In addition, *LG* students' increased awareness of the angle concept appeared richer in that they were much more likely to draw nonprototypical examples of angles.

LG students' descriptions of an angle emphasized the concepts of rotation and, to a lesser extent, bending more than those of control students did. Similarly, in the interviews, the *LG* students' descriptions of differences between their angle drawings demonstrated the learning of both the concept of angle and of the measure of angles for the purpose of classification.

Similarly, *LG* students outperformed controls on drawing a larger angle. Here, kindergarten, third-grade, and especially fourth-grade *LG* students seemed to benefit the most. The *LG* students at every grade scored substantially higher than controls on questions that measured knowledge about the amount of turn. Again, the interview results were consistent with this finding. Our results indicate that measure of turn is a concept that can be learned by students in any grade of elementary school. *LG* students outperformed control students in judging the size of angles at Grades 2 and higher. These tasks involved difficult distractors, which demanded, for example, that students differentiate between greater angle measure and greater side length. Most of the increase in the formal categories of explanation occurred among intermediate-grade students.

Many of these benefits were hypothesized to result from an emphasis on paths. We have also investigated possible criticisms of using the path perspective. For example, studying paths could lead to a confusion between measures of the amount of turn along the path and the measure of the angle. Our research to date indicates that this need not be so, even for young students, under proper instructional conditions.

There was no difference between *LG* and control groups on angle measure estimation. In a similar vein, performance of *LG* students was moderate on unit-test items of this type. In comparison, differences between groups were evident on tasks

that asked students to apply knowledge of angle measure in problem-solving situations. *LG* students were far more capable than control students of finding missing angle and side measures and of determining measures of turns. (Although performance was just over 50% correct, the unit-test items provide confirmation.) Also, *LG* students were substantially more capable of determining the measure of angles in word problems (for the boat item, substantial gains relative to controls were found for fifth and sixth graders but not for fourth graders; students in all grades from 2 to 6 performed better than controls for the more tractable wire item). Finally, knowledge of the total amount of turtle turning for polygons (shown on unit-test items) was somewhat greater than knowledge of the sum of the interior angles of polygons. This is reasonable, given that directing the Logo turtle emphasizes only the former. Yet the lack of large differences between the scores of *LG* students on items involving internal and external angles indicates that *LG* effectively facilitated students' development of both of these two concepts, with no detriment to either. Classroom observations reveal that personal experiences in *LG* (e.g., as opposed to only demonstrations) are essential in students' development, and differentiation, of these two concepts.

In summary, compared to the traditional curriculum, *LG* had only moderate positive effects on the students' ability to identify and draw angles or to estimate angle size. These results are consistent with those of Kelly (1986-87), who reported that benefits may not emerge until after more than a year of Logo experience. The *LG* curriculum that is the focus of this monograph had large facilitative effects, however, on students' conceptual differentiation between turns and changes of position, on the elaboration and sophistication of knowledge of angle and its measure, on knowledge of amount of turn, and on the application of knowledge of turn and angle measure in problem-solving situations (cf. Clements & Battista, 1989; du Boulay, 1986; Frazier, 1987; Kieran, 1986a; Kieran & Hillel, 1990; Olive et al., 1986). Probably because *LG* explicitly taught the relationship between amount of turn and measure of angle produced, students gained knowledge of both these measures as well as their relationship.

Integration of information from all data sources allows us to hypothesize that the development of the students' notion of the turn along a path moves through several steps. We conjecture (albeit post hoc) the following seven increasingly sophisticated, conceptual constructions.

1. Students construct notions for two distinct types of changes of state—change in heading (caused by the RIGHT and LEFT commands) and change in unidirectional displacement (caused by the FORWARD and BACK commands). Indeed, young students tend not to make any distinction between these two—often, they believe that a RIGHT command actually moves the turtle to a location to the right of its present location.

2. Students seem to view the turn commands (RT and LT) merely as changes in heading; they give little attention to the actual turning motion. They often issue repeated turn commands until the turtle is facing the desired direction.

3. To students, the turn motion and direction become salient. They recognize that if they have turned the turtle the wrong way, they can compensate for this by turning the turtle in the opposite direction. However, students act as if they believe that the direction of a turn that moves the turtle to a desired heading depends only on the final destination and not on the turtle's current heading. For example, to turn a turtle to a heading of 90° ("east"), these students would issue a RIGHT command, even if the turtle were at a heading of 180°.

4. Students recognize that the direction of a turn that moves the turtle to a desired heading is dependent on the turtle's heading, but they are unable to determine this direction without physical aid.

5. Students can determine the direction of a turn by modeling with their bodies. However, they seem unable to do so internally.

6. Students mentally project themselves into the place of the turtle. This is often accompanied by some physical movement—the turn of the head or upper body.

7. Students examine the turtle's present heading and desired headings. From these, the students determine the direction of turn by directly apprehending a rotational vector. In using this approach, students take a more extrinsic (rather than intrinsic) perspective.

In a follow-up study, we found that above-average, fourth-grade students did not show the initial difficulties with turns. However, we did confirm the constructions of Steps 4 to 7 again (Clements & Burns, 2000).

PATHS

The *LG* curriculum was designed to develop and build on ideas about geometric paths. Therefore, it is not surprising that students in this curriculum substantially outscored control students on path items. There was some indication that younger *LG* students made greater gains in this domain than older *LG* students, but students at all grades outperformed control students. Unit tests confirmed these results, with accuracy in drawing paths to fit given descriptions ranging from 72% to 95%, with most averages in the 80% to 90% range. Thus, the targeted path concepts seemed to have been learned well by students in all grades.

Items that required interpreting or producing Logo turtle commands to match given paths were more difficult, with accuracy varying widely depending on the complexity of the task and grade level (Grade 5–6 classes averaged in the 75% to 85% range). The complexity of mapping geometric figures to corresponding *LG* commands and vice versa, especially in the static and less familiar medium of paper and pencil, should not be underestimated. Nevertheless, more *LG* students than control students were able to use what they had learned to provide complete and accurate descriptions of the process of creating a two-dimensional figure.

Also, observations substantiated that it is natural for at least some students to view shapes in terms of paths. For example, when asked to tell whether two figures were the "same shape and same size" and then asked to explain what they would do "to

show someone that you are right," many young students that were interviewed responded in a path-oriented manner. Shown two congruent shapes resembling mittens (one the flip image of the other), one kindergartner said, "I would show them all around like this (traces the shapes with the fingers of each hand, simultaneously tracing corresponding components). They're the same, but they're just facing different ways."

SYMMETRY

Logo Geometry students scored significantly higher than control students on posttest measures of symmetry, both on tasks asking them to draw all the symmetry lines for given figures and those that asked them to draw the "other half" of a figure to create a symmetric figure. For the latter, there was a tendency for these effects to be particularly strong for young (kindergarten) *LG* students. In addition, the results were largely due to a greater number of *LG* students drawing completely correct figures.

LG students did not differ substantially from control students in the number of incorrect symmetry lines drawn. Indeed, evidence of overgeneralization by *LG* students regarding diagonal lines of symmetry and, to a lesser extent, lines of symmetry for figures that have only rotational symmetry should be addressed in instruction. In contrast, there was a clear and decisive positive effect of *LG* regarding correct lines of symmetry drawn.

Unit-test items confirmed these results. *LG* students were highly accurate in identifying whether a dotted line drawn on a figure was a line of symmetry (89% to 95%). Across Grades K–6, the most difficult items involved concave pentagons and parallelograms, where in each case the line shown was not a line of symmetry. Students performed somewhat less accurately when asked to draw all the lines of symmetry onto given figures (grade-level means on a two-point scoring system ranged from 1.46 to 1.64). The most difficult figures were those with more than two lines of symmetry (including all figures that were also those with oblique lines of symmetry) and the parallelogram, possibly due to the latter's rotational symmetry. Easiest were certain figures whose lack of symmetry was visually salient (e.g., an obtuse triangle for Grade 4–6 students). Finally, students were asked to draw the "other half" of one or more given figures to create a symmetric figure. This was a difficult task for kindergartners and moderately hard for those in the primary grades. Indeed, only the scores in Grades 5–6 were high. Most students, however, had the correct number of segments placed approximately in the correct position; achieving a high score required no substantive distortion.

At every grade in each school, the mathematics curriculum emphasized symmetry. Therefore, students above the kindergarten level had received increasing amounts of experience with symmetry before the study began, and control children studied the topic extensively during the course of the study. Nevertheless, *LG* students made significantly greater gains, which we attribute to several days of *LG* explorations. These explorations allowed them to build a measurably stronger

conceptualization of symmetric relations. Writing *LG* commands for the creation of symmetric figures, testing symmetry by flipping figures using *LG* commands, and discussing these actions apparently encouraged students to build richer and more general images of symmetric relations (possibly to the extent of overgeneralization in some instances) and to reflect on the construction of symmetric figures and abstract the properties of symmetry. Students had to abstract and externally represent their actions in a more explicit and precise fashion for the *LG* activities than they did in other activities, such as the freehand drawing of symmetric figures.

These findings support a related, later study, which added a special comparison group (Johnson-Gentile, Clements, & Battista, 1994). In the 1994 study, the *LG* group used the same curriculum as the *LG* students of the study reported in this monograph and similarly for the control group. A new treatment group created in 1994, which used manipulatives and paper-and-pencil versions of the *LG* activities on motions and symmetry, matched as closely as possible. This was designed to ascertain the specific contributions of the computer, per se. These noncomputer experimental students, for example, used paper and pencil and Mira for drawing and checking instead of using the *LG* MIRROR program.

Interviews revealed that both experimental groups, especially the *LG* group, performed at a higher level of geometric thinking than did the control group. Both experimental groups outperformed the control group on immediate and delayed posttests. Although the two experimental groups did not significantly differ on the immediate posttest, the *LG* group significantly outperformed the noncomputer experimental group on the delayed posttest. Investigation of individual items suggested specific effects on symmetry concepts. *LG* group scores increased and noncomputer experimental group scores decreased on tasks involving figures that had more than one line of symmetry or on parallelograms with no lines of symmetry. *LG* students appeared more likely to identify horizontal and oblique lines of symmetry and less likely than students in the other two groups to be misled by the rotational symmetry of the parallelogram. This suggests that differences between the groups were on those items that required students to resist applying intuitive visual thinking and to apply analytical thinking in a comprehensive manner.

CONGRUENCE

LG students performed better than control students on congruence items, not so much by identifying whether pairs of figures were congruent as by justifying their answers. Large gains on the justification score support the hypothesis that *LG* helps students conceptualize or formulate and communicate reasons for their mathematical decisions. This suggests a movement away from van Hiele Level 1, visual thinking, toward Level 2, descriptive/analytic thinking. Interestingly, the largest difference was for those questions with congruent pairs of figures. This difference suggests that *LG* helped students more on questions in which describing a difference (e.g., "this one is bigger") could not be used as a springboard to justification. Instead, a method (e.g., using geometric motions) was needed to describe a way

to "show" congruence. However, because these questions used more complex figures and motions, this conjecture must be evaluated in future research. Unit tests confirmed more than 90% accuracy on identification of congruent and noncongruent pairs of figures for the *LG* group in all grades. The most difficult items were those that required mental transformations of figures (rotations or reflections). But the impact of these difficult items decreased substantially in the intermediate grades.

GEOMETRIC MOTIONS

Because of the lack of systematic instruction of motion geometry in most control classrooms, the pre-post test in this study did not include items directly assessing this area of knowledge. Unit tests revealed that students in Grades K–1 identified single motions accurately, with many identifying sequences of motions as well. Students in Grades 2–6 identified single motions and flip lines and turn centers with increasing accuracy (74% to 91%). The most difficult motion to identify was a 180° rotation with a turn center not located on the figures. More difficult than identifying motions was identifying turn centers (especially those not on the figures) and flip lines for figures that were widely separated. These difficulties diminished as the grade level increased. Students in Grades 5 and 6 were administered two additional types of tasks. First, they were required to accurately draw an image from a preimage and a motion. The most difficult items were those involving turns, especially those with the turn center located at the top of the figure. The second type of task assessed knowledge of the relationships between motions and congruence. Most students understood that motions can determine congruence and that a sequence of motions can be found to move one congruent figure onto another. About half, however, may not have understood that slides and turns are not sufficient for this purpose. Before this conjecture can be accepted, more research is needed, since some students simply may have misinterpreted the item (e.g., as saying, "If two figures are congruent, there is a sequence of slides and turns that will move one onto the other, *accepting that slides and turns are the requisite motions*"). Interviews revealed that geometric motions were more salient for the *LG* students. They tended to sort by flips, slides, and turns in a more spontaneous, more frequent, manner.

In addition, the van Hiele level for motions increased more for the *LG* students, especially those from Site 1, than for the control students. Also, as predicted, these levels were equal to, or lower than, the van Hiele level for shapes for most students. For some reason, their thinking about motions may have been more natural than their thinking about shapes. Do these students possess high spatial ability? Or, are these students who had their shapes level decline from interim to the posttest? These students operated at the higher level and they scored higher on motions because they were tested immediately after the Motions strand. Therefore, forgetting did not really affect their motions performance. Finally, data suggest that the study of motions at the end of the *LG* curriculum may have caused some students to obtain

a lower van Hiele level on the shapes (triad) task, possibly because the motions unit made reasoning on the basis of visual transformation more salient. This possibility should be tested in future research. If true, it may suggest that a different order or nature of the presentation of these topics is warranted. In any case, these data constitute another indication that the students' movement from the visual to the descriptive/analytical level was tentative and fragile rather than strong.

Because the control groups in the main study did not have comprehensive experience with motions, we conducted a separate study to check these results, as described previously (Johnson-Gentile et al., 1994). Two experimental groups received the *LG* lessons on motions. However, one used manipulatives and paper and pencil to duplicate as closely as possible the computer activities of the *LG* group. These noncomputer experimental students worked with figures identical to those from the *LG* ROCKET program but they worked with figures that had been duplicated on acetate. To turn their rockets, they placed a pencil point at the turn center and used a protractor to determine the amount of turn. For this 1994 study, the control group received traditional textbook geometry instruction.

The three groups did not differ significantly on pretest geometry achievement, but both experimental groups significantly outperformed the control group on immediate and delayed geometric motions posttests. In addition, the *LG* and the noncomputer experimental students substantially outperformed the control students on tasks measuring levels of geometric thinking. This substantiates the conjecture that the curriculum is effective. Comparing the two experimental groups, *LG* students were more likely to forgo visual strategies and use Level 2 descriptive/analytic strategies for solving geometric problems. The *LG* group's scores significantly increased from the immediate to the delayed posttest, whereas those scores of the noncomputer experimental students significantly decreased, resulting in a significant difference in favor of the *LG* group on the delayed posttest. In a similar vein, scores of the *LG* group increased on items requiring the identification of motions and their properties (i.e., turn centers).

Although caution must be used in interpreting these post hoc analyses, they do suggest that significant differences on the delayed test can be attributed to the *LG* students' construction of higher-level conceptual structures; that is, *LG* students analyzed components and properties of geometric motions (van Hiele Level 2). Thus, this separate study revealed no evidence that use of the *LG* motions curriculum helped children perform tasks at Level 1 of geometric thinking better than control instruction; that is, such use did not appear to facilitate the ability to draw symmetric figures, draw figures given specified motions, or apply motions to problems such as sorting tasks. There is, however, evidence that the use of Logo facilitated students' transition to Level 2 thinking. *LG* students were more likely than students in the other two groups to describe the properties of geometric motions and symmetry constructs and more likely to solve problems involving these constructs at a more abstract and analytical level. This adds to the body of evidence supporting the hypothesis that Logo experiences can help students become cognizant of their mathematical intuitions and facilitate the transition from

visual to descriptive/analytic geometric thinking not only in the domain of two-dimensional figures but also in the domain of motion geometry (Clements & Battista, 1989, 1990, 1992b; Edwards, 1991; Gallou-Dumiel, 1989; Lehrer & Smith, 1986; Olive et al., 1986; Olson et al., 1987). This is significant because, even in secondary school, students may be at the lowest two van Hiele levels in transformation geometry, showing limited use of precise language and little knowledge of properties (Soon & Flake, 1989).

PROBLEM SOLVING

Both the "Turtle Deliveries" and "Equilateral Triangle/Regular Polygon" episodes illustrate important components of the environment that we have tried to create in our curriculum. Students are constantly trying to make sense of mathematical ideas. They are constantly reasoning and reflecting on that reasoning. They are constantly involved in mathematical problem solving. Indeed, we feel that these episodes portray students who are actively involved in instruction that effectively promotes the development of problem-solving ability. Silver and Thompson (1984) state that evidence for effective instruction in problem solving should not be restricted to counting the number of problems that students solve correctly. Instruction can also be judged as effective in this regard "if it produces students who spend more time in analyzing a problem's statement, search for alternate solutions to a problem, have greater confidence in their ability to solve problems, or show increased willingness to attempt difficult problems that may take a long time to solve. In fact, the benefits of such instruction may be more important than merely increasing the number of correct solutions to a narrow domain of problems" (p. 541). We believe that the episodes of students' activity with our curriculum above clearly demonstrate the type of instructional environments called for by Silver and Thompson and the NCTM *Standards*.

ISSUES OF EPISTEMOLOGY, MOTIVATION, AND IMPLEMENTATION

One consequence of *LG*'s problem-centered, exploratory approach is that we observe progress in the areas of reasoning and mathematical epistemology. That is, students' beliefs about and methods for establishing mathematical truth become more sophisticated. Early in the curriculum, students who were asked to justify their reasoning or responses often said, "That's what our teacher told us last year." Younger students are mute or say, "It just is" or "It looks like it." Later in the curriculum, they move toward empiricism by saying, "Look, I can show you," or "That's a square ... it's just tilted. I can prove it by measuring the angles."

Finally, some of the older students begin to reason on the basis of the network of relations between the properties of figures, to use van Hiele's terms. They begin to organize propositions, to string them together in sequences. For example, they reason that a "tilted square" is a square because the procedure used to draw it incorporates the defining properties of a square—or that every square must be a

rectangle because every square can be produced with a rectangle procedure just by making the two inputs for the sides the same.

Thus, observations of students show that they are moving from (a) nonanalytical or authoritarian-based notions of knowledge, to (b) more autonomous, empirical bases for establishing truth, and finally to (c) knowledge as reasoning about mathematics as a system.

Also, we believe evaluation results support the claim that that *LG* is consonant with the National Research Council's (1989, p. 6) recommendations that "students actually construct their own understanding based on new experiences that enlarge the intellectual framework in which ideas can be created."

Another important point we wish to mention is that all of the students in the project have been motivated by the curriculum. They enjoy working with the computer. They like playing with the new mathematical concepts. One group of students made large paths on the playground with chalk, then talked about them as they were viewed from their third-story classroom. In one inner-city school, the students talked for weeks about paths to different locations within the school. "I can get to the office by going left 90, forward 200, and left 90." In this same class, the students were also highly motivated and competent with the Motions strand. One day, the teacher was extremely pleased with the students' performance. She emphasized to them the value of working cooperatively. But, more important, she talked about how they had succeeded on a difficult task and how valuable it is to persist in problem solving; she emphasized to the students that they could solve problems if they persisted. This teacher felt, as do many teachers in the inner cities, that such students desperately need to be successful in mathematics and other academic pursuits. Success in solving problems in such courses as mathematics builds self-confidence. Many of these teachers feel that inner-city students get very little encouragement at home. And this particular teacher believed that our curriculum had provided a vehicle for students to learn how to be confident, which is more important to learn than the mathematics content.

In addition to our specific findings regarding *LG* students' achievement of the curriculum goals, our research has demonstrated that achieving these goals is more likely if both curricula and instructional approaches have four general characteristics, as follows. They should:

1. Systematically facilitate students' progression from basic intuitions and simple concepts to higher levels of geometric thought. Students should "put the pieces together" in a profound sense.

2. Rely on substantive exploration of ostensibly simple problems to promote deep and lasting conceptual learning. Students should search for relationships and meaning.

3. Help students develop the ability to move away from authoritarian bases of knowing ("the teacher told me") to autonomous bases ("I know because I can build it, measure it ..." and ultimately to the ability to "reason it out").

4. Allow children to construct geometric ideas actively. Geometric learning experiences should stress action in three related ways.

- *Include physical activity*—such as children walking, drawing, and tracing shapes;
- *Emphasize geometric shapes as active entities*—either formed by the action of the turtle moving along a path, or transformed by motions;
- *Include reflective activity*—in planning procedures, reflecting on those procedures, posing and solving problems, and considering the ideas of oneself and others.

Many research questions remain to be answered. Because the teachers helped develop the curriculum and because the number of sites was limited, our analyses could not adequately address questions such as the following. How do the hierarchically-leveled variables, including site, years of experience, teacher, and curriculum, interact? Are there significant gender differences and interactions (Noss, 1988)? Regarding the effects of teacher experience, we have mixed evidence. Overall, students from Site 1 *LG* and Site 2 *LG* classrooms did not differ in achievement, although the Site 1 *LG* teachers had more experience with the *LG* curriculum. However, there is a suggestion that such teacher effects may have been present on tasks that required complex geometric reasoning.

On a task needing such complex reasoning about the definitions of quadrilaterals, Site 1 *LG* students maintained their advantage compared to the Site 1 control students as the complexity of the task increased; however, the corresponding advantage for Site 2 students diminished. Students of teachers less experienced with the curriculum made smaller gains, especially on more complex, high-level tasks. Another way of stating this is that students whose teachers had more than a single experience teaching the curriculum made more substantial gains on complex tasks. Thus, our findings for teacher experience are neither simple nor certain. Individual teacher abilities appear to be more important than one versus two year's experience teaching the curriculum, although there is a suggestion that such experience may correlate with student gains on complex tasks.

LOGO GEOMETRY

Overarching positive effects of *LG* can be summarized simply. One main effect was in increasing students' ability to describe, define, justify, and generalize geometric ideas. Students' greater explication and elaboration of geometric ideas within the Logo environments appear to facilitate their progression to higher levels of geometric thinking. For example, in programming the Logo turtle, there is a need to analyze and reflect on the components and properties of geometric shapes and to make relationships explicit. In addition, the necessity of building Logo procedures encourages students to build understandable, implicit definitions of these shapes. Students construct more viable knowledge because they are constantly using graphical manifestations of their thinking to test the viability of their ideas. There is also support for the linkage of symbolic and visual external representations. The

students' use of Logo to negotiate between the two has been identified as a main source of learning (cf. Hoyles & Noss, 1987a). Finally, the programming language permits students to outline and then elaborate on and correct their ideas.

In a similar vein, there was a positive effect on the students' flexible consideration of multiple geometric properties. As we previously hypothesized (Clements & Battista, 1994), computer environments can allow the manipulation of specific screen objects in ways that assist students to view them as geometric (rather than visual/spatial) and to recognize them as representatives of a class of geometric objects. The power of the computer is that students simultaneously confront the specific and concrete with the abstract and generalized (as represented by the Logo code). Students treat a figure both as having characteristics of a single shape and as one instance of many such figures. (This is closely related to the notion of "situated abstraction," Noss & Hoyles, 1992.)

Logo environments appear to demand and thus facilitate precision in geometric thinking. In contrast, there is imprecision when students work with paper and pencil, and they can be distracted by the actual effort of drawing. The need in Logo environments for more complete, exact, and abstract explication may account for the students' creation of richer concepts. That is, by using Logo students have to specify steps with thorough specification and detail to a noninterpretive agent. The results of these commands can be observed, reflected on, and corrected; the computer serves as an explicative agent. In noncomputer manipulative environments, a student can make intuitive movements and corrections without explicit awareness of mathematical objects and actions. For example, even young children can move puzzle pieces into place without conscious awareness of the geometric motions that can describe these physical movements. In noncomputer environments, attempts are sometimes made to promote such awareness. Still, descriptions of motions tend to be generated from, and interpreted by, the physical action of the students (Johnson-Gentile et al., 1994). This interpretation of the results is consonant with previous research indicating prolonged retention and continuous construction of early Logo-based schemes for geometric concepts (Clements, 1987).

Finally, computer and classroom environments that promote a problem-solving approach to education, as *LG* does, appear to have benefits for the development of both mathematical concepts and processes (e.g., reasoning, connecting, problem solving, communicating, and representing). They seem especially beneficial for developing student competence in solving complex problems. Further, because students test ideas for themselves, computers can aid them to move from naive to empirical to logical thinking and encourage them to make and test conjectures.

In summary, LG places as much emphasis on the spirit of mathematics—exploration, investigation, critical thinking, and problem solving—as it does on geometric ideas. We believe that it has the potential to develop valid mathematical thinking in students. As a final example of this, consider first-grader Andrew. At the final interview, he was quite sure of himself. When asked to explain something he thought clearly evident, Andrew would always preface his remarks with an emphatic "Look!" On one item, he was asked, "Pretend you are talking on the tele-

phone to someone who has never seen a triangle. What would you tell this person to help them make a triangle?" (PP1.10, Table 32).

Andrew: I'd ask, "Have you seen a diamond?"

Interviewer: Let's say that they said "yes."

Andrew: Well, cut out a triangle. [Pause.] No, I made a mistake.

Interviewer: How?

Andrew: They have never seen a triangle. Well, cut it off in the middle. Fold it in the middle, on top of the other half, then tape it down, and you'll have a triangle. Then hang it on the wall so you'll know what a triangle is!

Interviewer: What if they said they hadn't seen a diamond?

Andrew: Make a slanted line over, then another slant the other way down, then another slanted line up, then another slanted line to the beginning.

Interviewer: [Thinks he is trying to describe a triangle] What?

Andrew: [Repeats the directions. Then ...] That's a diamond. Now, do what I told you before!

Andrew had done what mathematicians are so fond of doing: He had reduced the problem to one that was already solved! At the end, he asked, "Will this test be on my report card? 'Cause I'm doing really good!" Throughout the interview, it was apparent that Andrew was sure of his own reasoning and knowledge from his experience. Although Andrew is not typical of students in our project, it is important to note that students such as Andrew may later become mathematicians, scientists, and engineers. Andrew had been reflecting greatly on the ideas in the curriculum and relished the opportunity to discuss them so that he could demonstrate the results of his thought.

STUDENTS' KNOWLEDGE OF GEOMETRY BEFORE INSTRUCTION

Examining assessment items before any instruction (besides traditional instruction from previous grades; i.e., PP pretest scores) provides a portrait of the geometric knowledge of 1,605 elementary school students in two states (because there was only one classroom, we omit the seventh-grade data from this discussion). These data have already been presented in tables; in this section, we focus on particular findings.

Consistent with previous research reviewed in chapter 1, students' performance after traditional instruction was not high nor was much learned from one grade to the next. Consider students' scores on the triangle identification task (possible score, 14): 9.02, 9.03, 9.70, 10.07, 10.28, 11.47, and 11.34, for K–6, respectively.[12] Corresponding scores for the rectangle task (possible score, 15) were 9.47, 9.15, 9.12, 9.76, 9.67, 10.21, and 10.18; for the square task (possible score, 15), 12.10, 13.54, 13.13, 13.44, 13.28, 13.77, and 13.66. These are shape-recognition tasks. Besides not displaying mastery at any grade level (except for the simplest case of

[12] Recall that these are the averages of the pretest scores from the original tables.

squares, for which near 90% correctness scores are arguably close), these elementary school students did not improve substantially across the grades for any item.

We learn more by examining individual items, with triangles providing the most fecund case. Virtually all students correctly identified the quadrilateral as a nontriangle. Most students, especially those older than kindergartners, could identify standard triangles in standard orientations (e.g., Shape 1 in Item PP/1.1; see Figure 1). Triangles in a nonstandard orientation (Shapes 6 and 8) were identified somewhat less accurately, with students' scores for these figures increasing slightly with age. Although most students responded accurately on these geometric figures, it remains questionable whether seven years of schooling should be necessary to reach acceptable, but not perfect, levels of performance.

Performance was lower on the figures with convex curvilinear sides (Shapes 3 and 7), and there was little sign of scores increasing with grade. On the two figures with concave curvilinear sides (5 and 14) performance was quite low. (The relatively high performance of the kindergarten classes on both of these figures is curious.) An inference is that traditional geometry curricula do not provide adequate experiences with such nonexamples nor with discussions of the necessary properties of triangles (e.g., straight sides). More consistent progression with age is seen for the concave quadrilateral (Shape 9). Performance was quite low for primary-grade students but increased to moderately high levels for upper intermediate-grade students. On the two open shapes (Shapes 4 and 13), students achieved moderate accuracy. Both mastery and grade-related increases were evident for Shape 13; the need to continue an oblique line segment over a greater distance may have led fewer students to "close" this figure. Provision for open "triangular" figures and discussions of closure are probably neglected in traditional curricula, even though research has established that this property is salient even to young children.

Students' ability to correctly identify Shape 11, an obtuse triangle at a nonstandard orientation, improved with grade, but even some older students continued to misclassify this figure. The nonsimple figures (10 and 12) may have confused students. Performance was relatively low and developmental trends were not clear. Again, these results confirm one major conclusion: Traditional geometry curricula are deficient. They do not provide a sufficient range of systematically varied exemplars and nonexemplars. Perhaps more significantly, they do not provide students with an adequate experience of discussing such figures and making decisions to classify them on the basis of a consideration of their properties.

In a similar vein, students had difficulty attributing properties to figures, with scores of 5.36, 4.21, 4.37, 6.23, and 6.13 for squares and 4.45, 4.13, 4.05, 5.46, and 5.08 for rectangles, for Grades 2 to 6, respectively (these are averages of the pretest scores in Table 6). Again, this shows moderate performance and small gains at best. Only a small minority of intermediate-grade students could identify what shape a closed figure could be when given a set of properties (except for the obvious choice; see Table 7).

Turning to a critical geometric object, the angle, we find that 29% of students could not provide any description of "angle" and 13% described angle as a tilted

line. On a simple angle-identification measure, scores out of a possible 14 were 7.57, 7.75, 8.17, 7.52, 9.17, 11.06, and 11.23 for Grades K to 6, respectively. For the single item of drawing an angle, scored 0 or 1, the respective means were 0.08, 0.20, 0.11, 0.18, 0.50, 0.89, and 0.84. In Grades 2–4, most errors in drawing an angle involved drawing geometric shapes and tilted lines. Thus, a majority of students could identify angles and a majority of intermediate-grade students could draw an angle, but many still had few ideas, or only nonmathematical ideas, about the angle concept.

For the single "drawing a bigger angle" item, means for Grades K–6 were 0.02, 0.02, 0.02, 0.02, 0.16, 0.43, and 0.51, respectively. For the "spinner" item assessing the estimation of the amount of turn, means for Grades 2–6 were (out of a possible 3), 0.08, 0.00, 0.21, 0.66, and 0.68, respectively. For multiple-choice estimation of the measures of drawn angles (out of 4), means were 1.50, 1.65, 2.36, and 2.94 for Grades 3 to 6, respectively. For calculating the measures of missing angles and sides (out of 7), means were 1.69, 2.37, and 2.53 for Grades 4 to 6, respectively. (Almost no primary-grade students could use arithmetic processes to solve geometric problems accurately; see Table 9. Only a small minority could solve more complex angle-measure problems, such as the "boat" and "wire" tasks.) Together, these means indicate that students in Grades 5 and 6 are learning about angle measurement, but they answered only half to three fourths of simple problems correctly and evinced much lower competence on more complex problems.

On drawing lines of symmetry, the students' scores (out of a possible 8) were 0.49, 0.30, 1.34, 2.36, 3.15, 4.42, and 3.85 for Grades K–6, respectively. On drawing the other half of a figure, means (out of a possible 2) were 0.01, 0.18, 0.52, 0.90, 1.04, 1.43, and 1.35 for Grades K–6, respectively. For symmetry, then, growth is more evident than for most other areas, although performance of even the oldest children has much room for improvement. The decrease from Grades 5 to 6 has no clear explanation and should be checked in future research.

Scores on identifying congruent figures (out of a possible 12) were 7.56, 8.01, 8.49, 8.29, 8.32, 8.52, and 8.58 for Grades K–6, respectively; for justifications, the scores (out of 6) were 2.71, 3.51, 3.92, 1.18, 3.77, 4.52, and 4.58. Thus, students show moderate competence in identifying congruent figures but do not improve substantially from kindergarten to sixth grade. Students do show improvement in justifications.

In summary, consistent with national and international assessments and other studies (Beaton et al., 1996; Carpenter et al., 1980; Fey et al., 1984; Kouba et al., 1988; Lehrer et al., 1998; Stevenson et al., 1986; Stigler et al., 1990), the performance in geometry of students who were given traditional instruction was low to moderate. Students are not learning much geometry from one elementary grade to the next. Results reported here confirm those findings and extend them across a range of topics and types of problems in elementary school geometry. Traditional geometry curricula are deficient (cf. Clements & Battista, 1992b; Fuys et al., 1988; Lehrer et al., 1998). Results presented here confirm this deficiency and indicate that such curricula do not provide (a) a sufficient range of systematically varied

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exemplars and nonexemplars, (b) adequate experience discussing geometric objects and their properties, or (c) rich experiences that include geometric reasoning and geometric problem solving. There is a need for more, different, and enhanced geometry education in every grade from kindergarten to sixth.

Chapter 5

New Development and Research on Logo and Geometry

by Julie Sarama

Although the research conducted with the Logo Geometry Project took years to complete, many of the lessons learned during the project spawned later software development and research. This chapter discusses these activities and related empirical research. For brevity's sake, this chapter only overviews these results, and thus the level of specificity and "grain size" are different from the previous chapters; references are provided to articles that provide full detail for the interested reader.

TURTLE MATH

The authors of this report were involved in another National Science Foundation project, following *Logo Geometry*, that developed the geometry and software for the *Investigations in Number, Data, and Space* curriculum with colleagues at TERC. Douglas Clements and Julie Sarama (previously Julie Meredith) created a new version of Logo that built upon the *LG* microworlds and research. This new version, called "Geo-Logo" in the context of the *Investigations* curriculum, was developed as a stand-alone product called "Turtle Math" (Clements & Meredith, 1994). In the chapter, we describe the development of Turtle Math, and the results of several studies on its use in learning mathematics. We begin with a description of the design principles.

Design Principles and the Turtle Math Environment

In the development of Turtle Math (Clements & Sarama, 1995), we (Julie Sarama and Douglas H. Clements) abstracted five principles from the extant research, including *Logo Geometry*. Using these principles we made specific design decisions. Figure 29 is an illustration of the Turtle Math environment, followed by a discussion of each of the principles.

1. *Encourage construction of the abstract from the visual and intuitive.* The nature of the interaction in certain environments promotes the connection of two types of external representations, formal symbolic and dynamic visual, supporting the construction of mathematical strategies and conceptions from initial intuitions and visual approaches. Turtle Math supports such connections in four ways. Consider

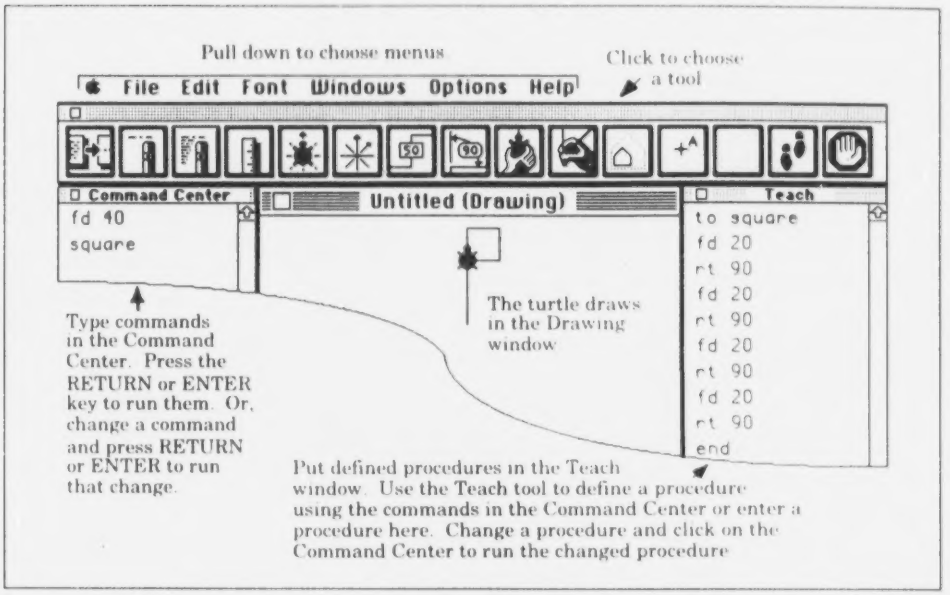
Pull down to choose menus
Click to choose a tool
File Edit Font Windows Options Help
Command Center
fd 40
square
Untitled (Drawing)
Teach
to square
fd 20
rt 90
fd 20
rt 90
fd 20
rt 90
fd 20
rt 90
end
Type commands in the Command Center. Press the RETURN or ENTER key to run them. Or, change a command and press RETURN or ENTER to run that change.
The turtle draws in the Drawing window
Put defined procedures in the Teach window. Use the Teach tool to define a procedure using the commands in the Command Center or enter a procedure here. Change a procedure and click on the Command Center to run the changed procedure

Figure 29.

the difficult topic of turns and angle measure. First, Turtle Math provides two commands, "rtf" (stands for right face) and "ltf," to help beginning students build on previous experience and appreciate turns qualitatively before encountering measurement in degrees. Second, Turtle Math provides measurement tools. The "line of sight" tool (fifth icon from the left in Figure 29) displays rays in 30° increments at the turtle's heading and position. An on-screen protractor called the *turtle turner* (sixth from the left in Figure 29) provides an exact measurement for any turn or angle, as shown in Figure 30 below. Third, the "label turns" tool (eighth icon from the left in Figure 29) allows students to view the measures of all turns on the

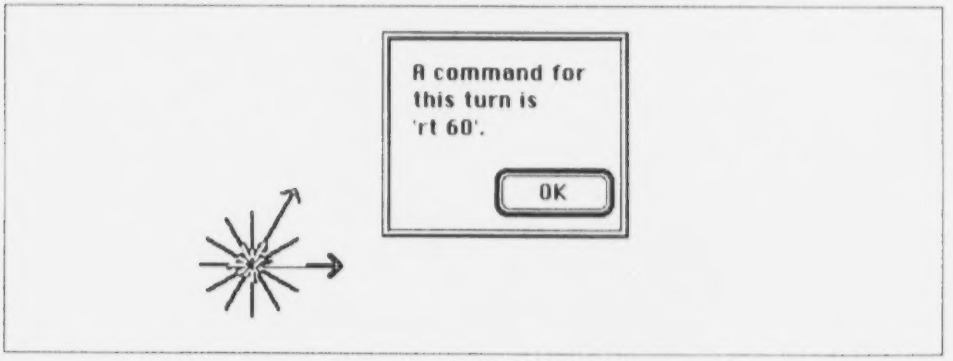

Figure 30.

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screen. Another way Turtle Math supports the growth of the abstract from the visual lies in its overall structure, described in the following section that deals with the second principle.

2. *Maintain close ties between external representations.* The nature of programming creates the need to make relationships between symbolic and visual external representations explicit. But students may lose the connection between the two. Turtle Math encourages immediate mode programming; that is, students type commands into the command center and see them immediately enacted on the screen. A tool (farthest left on the toolbar) transforms a set of commands into a procedure in the "Teach" window with one step. Further, any change of commands in either location, once accepted, is reflected automatically in the drawing. In this way, the command window code constitutes what we call *a proleptic procedure.* The Logo code in the command window stands halfway between traditional immediate mode records and procedures created in an editor, helping link the symbolic and visual representations. The structure of the Command Center—long and narrow to the side of the graphics screen instead of the traditional short but wide placement below this screen—permits the immediate inspection of more commands, which facilitates connecting symbolic and visual representations and pattern searching. Moreover, the admissibility of the code to modification encourages experimentation and exploration and scaffolds working with true procedures.

Another issue is the directionality of the symbol-visual connection. One of Logo's main strengths has been its support of linkages between visual and symbolic representations. One of its limitations has been in the lack of bidirectionality between these modes (Noss & Hoyles, 1992). That is, one creates or modifies symbolic code to produce visual effects, but not the reverse. Turtle Math provides a "draw commands" tool (ninth icon from left in Figure 29) that allows the student to use the mouse to turn and move the turtle, with corresponding Logo commands created automatically. A related, "change shape" tool (tenth icon from left in Figure 29) allows a student to manipulate figures on the screen and watch the commands that generated them change automatically. (See Figure 31 for an example of these tools in operation.)

Although the modification of the figure results in the modification of the code, the code should ideally be easy to modify as well. This leads to the third principle.

3. *Facilitate examination and modification of code.* Although computers have the potential to help encourage analytic thinking, research indicates that there may be little reason for students to abandon visual approaches in favor of analytic approaches unless they are provided with support for such analysis and are challenged with tasks whose resolution requires an analytical approach. Several of Turtle Math's tools provide such support. The Command Center interface already described helps students examine the sequence of commands reflected on the graphics screen, helps them modify these commands, and allows them to see immediate results from this modification. Tools for editing code (in addition to the "teach" tool already discussed) allow erasing (the second and third tools from the

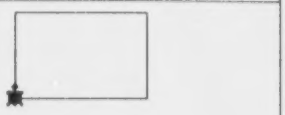
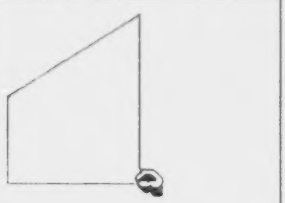
Using the "draw commands" tool, a student might continuously create first a turn (RT or LT) command.	LT 45	
Then a movement (FD or BK) command, and so on, until she clicks outside the window.	LT 45 FD 100	
Using the "change shape" tool allows a student to manipulate figures on the screen and watch the commands that generated them change automatically.	FD 50 RT 90 FD 75 RT 90 FD 50 RT 90 FD 75 RT 90	
In the example shown here, the student has first manipulated a side of a rectangle. As she slides the side, the commands change automatically and dynamically, as shown.	FD 50 RT 70 FD 80 RT 110 FD 50 RT 70 FD 80 RT 110	
She then manipulates a vertex. In this case, different commands are altered.	FD 50 RT 70 FD 80 RT 110 FD 77 RT 90 FD 75 RT 90	

Figure 31.

left erase one command or all commands in the command window, again with dynamic links to the geometric figure) and inspecting and changing commands (the "walking feet" icon—second from the right in Figure 29—is the "step" tool, which allows students to "walk through" any sequence of commands; each command is simultaneously highlighted and executed).

4. *Encourage procedural thinking.* Turtle Math provides a Command Center window as a proleptic procedure, tools to "walk through" the definition of a procedure, and automatic connections between changes in procedures and changes in drawings.

5. *Allow a domain in which to explore diverse areas of mathematics using a consistent metaphor.* The benefit of having a consistent way to talk and think about different areas of mathematics is that it can serve as a way for students to bring coherence to a varied mathematics curriculum.

Initial Field Test

The results of the first field test, involving several third-grade classrooms, are summarized here for four of the five principles (for full details, see Clements & Sarama, 1995). Insufficient data were collected for the fifth principle.

1. Turtle Math appeared to encourage construction of abstract concepts from visual intuitions. For example, intuitive commands, such as "right face," helped students appreciate turns qualitatively at first. Then, restricting turns to multiples of 30 and providing measurement and labeling tools scaffolded students' construction of measurement in degrees. Further, there was no evidence of confusion between turn and movement commands, though such confusion has been frequently reported in previous research.

2. Turtle Math helped children establish connections between external representations of mathematical ideas. The Command Center's shape and the dynamic link between the Logo code and the graphic drawing enhanced students' propensity and ability to connect symbolic and visual representations and to connect those representations to number concepts, especially when combined with the "energy level" feature of several activities. Children were highly motivated to reflect on the commands and their precise geometric effects, perhaps leading to an understanding of both the advantages and disadvantages of algorithmic solutions. Overall, the structure of Turtle Math, especially the Command Center, played a significant role in encouraging children to accurately estimate lengths, synthesize parts into wholes, apply arithmetic in situations that were meaningful to them, and reflect on path/command correspondences. In the Logo Geometry Project, we found that path/command correspondence had to be taught explicitly, and so an activity was developed for this purpose. With Turtle Math, no specific, separate instruction was given on this idea. The structure of Turtle Math and the associated activities alone resulted in all children using the idea in problem-solving activities, with no observed instances of confusion regarding the idea.

3. The ease of editing Turtle Math code and the dynamic linking feature facilitated students' reflection on, and modification of, their Logo code, which helped them experiment with different problem solutions (e.g., amount of turn). In some cases, however, these features may have allowed random trial and error without reflection. Ease of examination and modification led to spontaneous explorations. On one occasion, the teacher of one of the third-grade classrooms saw an opportunity to engage students in the mathematical activity of problem posing when discussing students' procedures for equilateral triangles. The class was discussing how the numbers changed for making bigger and smaller triangles.

Teacher: Great. We got the turtle to draw bigger and smaller equilateral triangles. Who can summarize how we did it?

Maurice: We changed all the forward numbers with a different number—but all the same. But the turns had to stay 120, 'cause they're all the same in equilateral triangles.

Chris: We didn't make the biggest triangle.

Teacher: What do you mean?

Chris: What's the biggest one you could make?

Teacher: What do people think?

Ronald: Let's try 300.

The class did.

Maurice: It didn't fit on the screen. All we see is an angle.

Teacher: Where's the rest?

Beth: Off here [gesturing]. Remember, we've got it set so that the turtle doesn't wrap around the screen.

TA: Let's try 900!

The student typing made a mistake and changed the commands to FD 3900.

Teacher: Whoa! Keep it! Before you try it ... class ... tell me what it will look like!

RN: It'll be bigger. Totally off the screen! You won't see it at all!

Jereb: No, two lines will still be there, but they'll be way far apart.

The children were surprised when the drawing turned out ... the same!

Teacher: Is that what you predicted?

Ronald: No! We made a mistake.

Maurice: Oh, I get it. It's right. It's ... just farther off the screen. See, it goes *way* off, there, like about past the ceiling.

The teacher challenged them to explore this and other problems they could think of.

Jereb: I'm going to find the smallest equilateral triangle.

Ronald: We're going to try to get all the sizes inside one another.

This episode also supports the notion that Turtle Math's structure supports the development of abstract ideas from students initial visual intuitions (principle 1). The Teach window's omnipresence (there is no need to invoke an editor or a "flip side" to view procedures stored in it), shape, and dynamic link to the geometry of the figures it produced were vital features in encouraging students to engage in exploratory mathematical activity.

4. Turtle Math supported the creation, revision, and use of procedures. For most students using regular Logo, procedure definition is limited to saving work (Hillel, 1992). Initially, this was true for the students in the study; however, the procedure definition soon transmuted or transformed into higher levels of procedural use, including explorations with procedures. Although fully supported by Turtle Math, the use of procedures as a programming technique was not expected in this novice third-grade class. However, all students defined procedures from the beginning and used subprocedures in defining superprocedures with understanding. Thus, the

Turtle Math structure scaffolded immediate mode into procedural planning processes.

Yearlong Field Test

In this study, Turtle Math was tested through a yearlong use in two fourth-grade classrooms (Sarama, 1995). Data were collected by using both in-class observations and teaching experiments and were then analyzed using qualitative methods. The nature of the students' mathematical explorations with Turtle Math provided additional insight into how the software promoted mathematical learning that was consistent with the design principles, which were operationalized into goal behaviors. The four Process Standards (connections, communication, problem solving, and reasoning) of the NCTM (1989) were similarly operationalized. Eventually, two overarching themes emerged across the design principles: motivation and facilitation.

The Turtle Math environment had a positive effect on students' motivation to engage in most goal behaviors defined by the design principles and the NCTM Process Standards. Of those related to the five design principles, Turtle Math had a neutral effect on students' use of procedures (Design Principle 4); motivated students to establish connections between representations, to reflect on and modify their code, and to use a consistent metaphor to think about mathematics (Design Principles 2, 3, and 5); and motivated and required students to construct abstract concepts from visual intuitions (Design Principle 1). Of the four main NCTM Process Standards, Turtle Math motivated students to make connections and to communicate (the latter only partially) and motivated and required students to solve problems and reason mathematically. The nature of the motivation is that the students are rewarded intrinsically for engaging in these behaviors. That is, the software does not explicitly tell them to engage in these behaviors, nor does it offer reinforcement. Instead, the students engage in these behaviors because doing so helps them meet the needs of the situation.

Turtle Math software facilitated the behaviors specified by the five design principles and four NCTM Process Standards. Facilitation was further classified into three subcategories: mechanical facilitation, conceptual facilitation, and facilitation of students' application and extension of mathematical ideas to situations beyond the Turtle Math environment and activities, per se. After analyzing the data and comparing the results to the initial field tests, it became apparent that the software facilitated the goal behaviors related to the principles and standards; however, again, this facilitation varied across goals and subcategories.

Principle 1. Turtle Math facilitated students' construction of the abstract from the visual. That is, the level of students' abstraction increased with the amount of time they worked with the concepts in Turtle Math. This was most evident with the concept of length, in which they internalized the grid provided by Turtle Math. In the first activity, these students used this grid to begin constructing a flexible mental ruler that helped them estimate various lengths when using different scales.

In the domain of turns, although the students did not use ltf and rtf, they built up knowledge of turns over time using benchmarks of 90° in their estimating. Because of the lack of instruction, the tools that were designed to facilitate the abstraction of turn concepts were not used to a great extent, except for the line-of-sight tool. Indeed, use of the line-of-sight tool, in addition to participation in the turn activities, was efficacious in facilitating the abstraction of turn measure. If the students are to extend these abstract concepts and be able to apply them to situations outside the computer environment, they need as many opportunities to do so as possible. Thus, it appears that extensive use of Turtle Math over time is necessary for students to generalize their abstracted concepts.

Principle 2. The structure of the Command Center connects the Logo code to the graphics window, thereby linking these external representations. Teachers and instructional activities, however, are important catalysts in order for the students to consciously establish that connection. Because the students in this study did not engage in the same type of activities as the students who were part of the initial field tests, they were slower to make the connection between the code and the graphics. Use of Turtle Math over time, however, did facilitate students' conscious awareness of these connections. Once they became facile with path-command correspondence, the students extended such relations to off-computer situations naturally. Thereafter, they could discuss a set of commands written on a chalkboard and decide what graphic figure the commands would produce and then talk about altering certain commands and getting a different graphic effect.

Principle 3. The environment supported the students in the mechanisms of examining and modifying Logo code. The long Command Center allowed the students to see several commands at one time. The ease of using merely the delete key or the erase-one tool facilitated simple editing. Although the step tool was rarely used, as the students became more proficient at programming, the structure of the Command Center alone, with its dynamic link to the graphics window, supported their debugging and geometric explorations. This structure became so natural to the students that they were able to modify code in off-computer situations, even though they were unable to immediately see the results of their changes. Thus, the software supported the conceptual examination and modification of code.

Principle 4. The goal of procedural thinking was facilitated at the mechanical level. If the students were asked to define a procedure, they simply clicked on the teach tool and did so. However, the students were thinking procedurally only at the lowest level (e.g., in order to save work). The single class session in which the students did begin to use procedures more conceptually followed the teacher's encouraging and modeling the use of the procedure and then initiating an activity that necessitated writing procedures. Overall, however, students were not observed using procedural thinking in a generalized sense at all during the course of this year. Thus, unless the teacher specifically and explicitly taught and modeled the use of procedural thinking, the features of the environment that were intended to support such thinking were not utilized.

Principle 5. Given an entire school year to work in the Turtle Math environment, students used the turtle metaphor to think about the situation on the computer even when the situation went beyond typical turtle geometry. They made connections between mathematical topics that were usually segmented by the school curriculum when they worked on Turtle Math projects that required knowledge of several different concepts (e.g., coordinates, turn measure, length estimation, fractions, and decimals). Also, the turtle metaphor was applied without teacher mediation to those situations outside the computer environment that evoked the students' experience on computer. During the course of this study, these off-computer situations were limited to geometrical settings.

The software similarly facilitated the four mathematical processes of problem solving, communication, reasoning, and connections that are recommended by the NCTM *Standards*. The following paragraphs briefly discuss how this facilitation occurred.

Problem Solving. Turtle Math facilitated problem solving at a conceptual level in that the students needed to hone their problem-solving skills to solve the variety of problems they encountered during the year. In other words, they tried their solution, evaluated the results, reevaluated their initial strategy, monitored how well things were going in general, and retested final solutions to be sure the original task was completed.

The experience of solving problems on the computer had a positive effect on the way the students solved problems when they were not on the computer. One of the teachers, Mrs. Smith, noted that her students, especially the better students, were far less likely to give up on problems after they had worked in Turtle Math. She noted that these students initially started the year accustomed to being "good at math," but then they had difficulty working through novel problems. In order to minimize this difficulty, Mrs. Smith presented her students with off-computer problem-solving situations throughout the year. Still, she became convinced that it was the computer that "took (their problem-solving competencies) one step further."

Communication. Turtle Math facilitated mathematical communication because it gave the students a language with which to express their mathematical ideas. Also, because the students worked in pairs, communicating about their ideas became an essential element of completing a project.

Reasoning. Similarly, the nature of Turtle Math's noninterpretive feedback, as well as the structure of its Command Center, facilitated mathematical reasoning. This reasoning also was observed during off-computer geometry lessons. Most of the teachers involved in this research project attributed the development of mathematical reasoning to working in the Turtle Math environment.

Connections. The NCTM Standard regarding connections, which is similar to our Design Principle 5 was also facilitated by Turtle Math; that is, the students worked on projects that required them to synthesize concepts and processes from

several domains of mathematics. These connections seemed to be maintained in off-computer discussions about those projects as well. The teachers commented that the students pointed out connections and remembered the activities they worked on more than the teachers did themselves!

In summary, Turtle Math effectively transformed these fourth-grade classrooms into mathematical learning environments of the type recommended by the NCTM's *Standards.*

LEARNING SPECIFIC MATHEMATICS TOPICS WITH "TURTLE MATH"

In three other studies, we (Sarama, Clements, and colleagues) investigated the development of specific mathematics knowledge and competencies in Turtle Math environments. Here we will summarize briefly the results of each study, then discuss the role of Turtle Math and the implications for teaching by considering the three studies as a whole.

One study examined students' development of linear measure concepts (Clements, Battista, Sarama, Swaminathan, & McMillen, 1997). We observed three levels of strategies for solving length problems, and we hypothesized that students pass through these levels during their development of length concepts. Students, operating at the first level, did not segment lengths and also did not connect the number for the measure with the length of the line segment. Rather, they applied general strategies, such as visual guessing of measures and naive guessing of numbers or arithmetic operations. These students tended to be identified by their teacher as those with low mathematical ability.

Strategies at the second level were most common among third graders. They drew hash marks, dots, or line segments to partition or segment (i.e., not maintaining equal length components) lengths. Two factors could have caused this. A turtle step is a small unit (1 mm or less on most monitors); moreover, students' experience is probably such that objects 100 units in length are substantial in size. These factors may have made the abstraction of the turtle step difficult for those who wished to assign numbers in a meaningful, quantitative manner. Therefore, these students marked off lengths in units that made sense to them, usually units of 10. They needed to have such perceptible units to quantify the length.

Students who used strategies at the third level, like those at the first level, did not use figurative partitioning (or had ceased using partitioning at some point). However, they did use quantitative concepts in discussing the problems, drew proportional figures, and sighted along line segments to assign them a length measure. Therefore, we assume that they had what Steffe calls "interiorized" units of length and had developed a measurement sense that they could impose mentally onto figures. These observations substantiate Steffe's argument that these students had created an abstract unit of length (Steffe, 1991). This is not a static image, but rather it is an interiorization of the process of moving (visually or physically) along an object, segmenting it, and counting the segments. When consecutive units are

considered a unitary object, the student has constructed a "conceptual ruler" that can be projected onto unsegmented objects (Steffe, 1991).

We hypothesized that once students at the first level have had sufficient physical measurement experience iterating and partitioning into units, they construct schemes that allow them to partition unsegmented lengths. Such second-level schemes are figurative; that is, students need to use physical action to create perceptual partitions. In solving problems, these partitioning schemes develop to include the constraint that equal intervals must be maintained. This constraint leads, in turn, to the construction of an anticipatory scheme, because the equal-interval constraint can be realized most efficiently when it is done in imagery, in anticipation, without forcing perceptual markings. At this point, third-level strategies emerge.

The second study investigated the development of turn-and-turn measurement concepts (Clements, Battista, Sarama, & Swaminathan, 1996). Turns were less salient for students than "forward" and "back" motions. Still, they evinced a progressive construction of imagery and concepts related to turns. In addition, they gained experience with physical rotations, especially rotations of their own bodies. In parallel, they gained limited knowledge of assigning numbers to certain turns, initially by establishing benchmarks. For some students, a synthesis of these two domains—turn-as-body-motion and turn-as-number—constituted a crucial juncture in learning about turns. Some common conceptions, such as conceptualizing angle measure as a linear distance between two rays, were not in evidence (although they are reported frequently, e.g., Clements, 1987; Cope & Simmons, 1991; Hoyles & Sutherland, 1989; Kieran, 1986a; Kieran et al., 1986). Our results supported the efficacy and usefulness of the instructional activities.

In a follow-up study, we found that above-average fourth-grade mathematics students did not show the same initial difficulties with turns. In fact, they synthesized and integrated the two schemes, turn-as-body-movement and turn-as-number. Further analysis has revealed a process of psychological curtailment in which students gradually replace full rotations of their bodies with smaller rotations of an arm, hand, or finger, and they eventually internalized these processes (Clements & Burns, 2000).

The third study focused on the development of concepts of geometric figures (Clements, Sarama, & Battista, 1998). Work with Turtle Math facilitated students' conceptualization of the properties of geometric shapes as well as the students' connection between these geometric properties, measurements, and number conceptions. Both the researchers and teachers were reminded of the depth and long-term development of many ostensibly simple ideas, such as "shape" and "triangle." Many students' ideas remained a combination of those developed through the class work and from previously developed concept images. However, certain ideas in the unit, such as that of a simple, closed path and turtle motions (FD and BK) and turns, seemed to be significant organizing principles for the students. Our results support the notion that students should become explicitly aware of these principles and then of other properties of geometric figures that are less immediately salient to them, such as equality of turns and angles. They

need to apply these ideas to figures in order to determine whether or not the latter are members of specific geometric categories.

IMPLICATIONS

There are several implications that can be drawn from these studies regarding the design of mathematics curricula and particularly of geometric software activities. In general, we found that computer environments such as Turtle Math can encourage and support the development and use of mathematical concepts and processes. In fact, benefits can be attributed to specific characteristics of the Turtle Math computer environment.

First and most fundamentally, turtle geometry constitutes a context in which the components and properties of geometric shapes are critical. For example, in our studies students had to enter commands that specified the type of component (e.g., line segment or angle) and its measure. In this way, Turtle Math allows students to distinguish between attributes that are and are not defining properties of shapes, such as orientation.

Second, the computer provides intrinsic, nonevaluative, and meaningful feedback that is instrumental in motivating students to reflect on ideas about shape, length, and turns. In many instances in our studies, computer feedback led students to relate side lengths or angle measures to the properties of geometric figures. Turtle Math feedback consists largely of the graphic results of running Logo code precisely, without human interpretation. For example, many students would use Turtle Math to explore iterations of turns (RT 90 RT 90; or RT 30 RT 30 RT 30) and thus build a (superordinate) unit for measuring rotation. Students were less confident about the results of combining inputs to turn commands than combining inputs to FD and BK commands. Initially, most students gave no indication that they were combining rotational units, per se (i.e., quantities). Rather they appeared to operate on numbers (without quantitative grounding), use the result of that numerical operation as the input to a new turn command, and check the resultant final heading of the turtle. Because students received this immediate feedback from the computer, they were able to give quantitative meaning to this combining operation.

Third, the computer context is motivating. Several students, categorized by their teachers as the "lowest ability" students in mathematics, often solved paper-and-pencil problems with visually-based guessing. However, these same students deliberately used arithmetic when working in Turtle Math. Teachers had to stop one pair of boys who were fighting between themselves over who would get to do the arithmetic to compute the missing length for a rectangle.

A fourth advantage of Turtle Math is its flexibility and dynamic connections between the symbolic and graphic. The geometric setting provided both motivations and models for students' thinking about geometry, number and arithmetic operations, and their interaction. The motivations included game settings and activities that allowed students to create their own pictures. The models incorporated length and rotation as settings for building a sense of both numbers and oper-

ations on numbers, with measuring and labeling tools supporting such construction. Conversely, the numerical aspects of the measures provided a context in which students had to attend to certain properties of geometric forms. The measures made such properties (e.g., opposite sides equal) more concrete and meaningful to the students. The dynamic links between these two domains, structured in the Turtle Math environment (e.g., a change in code was automatically reflected in a corresponding change in the geometric figure), facilitated students' construction of connections between their own number and spatial schemes.

Fifth, tools based on research and fine tuned through field testing can encourage students' mathematization. For example, Turtle Math's slow turns were noted by students and helped them build dynamic imagery for rotations. The measurement and labeling tools encouraged them to eschew guessing when they were required to use such mathematical processes as measurement and analysis. Unlike regular Logo, commands in Turtle Math that are the same stay the same and do not have to be reentered. More important, those that are changed are not executed as an additional command affecting the graphics; instead, the figure changes to reflect the change in the commands. This feature helped students encode contrasts between different commands. Therefore, these studies, then, add to the literature indicating that enriching the primitives and tools available to students facilitates students' construction of geometric notions and increases their use of analytical, rather than visual, approaches (Clements & Battista, 1992a; Clements & Sarama, 1995; Kynigos, 1992, 1993).

As we mentioned before, there are also implications that can be drawn from the studies regarding curriculum design. In the following paragraphs, we discuss seven implications that result from the studies described in this chapter. First, although students use some of the aforementioned tools "naturally," they must be encouraged to use others. For example, although the step tool is useful for debugging or tracing large procedures, students need to be taught how to use this tool when working on the computer. A short lesson showing how the step tool works will not result in students using this tool to its fullest potential. If the teacher does not continue to model the use of it and other tools, these tools may be forgotten after an initial exploration.

Second, curricula and teachers should encourage connections throughout the mathematics curriculum. For many students, connections between geometric forms and numerical ideas are tenuous at best, even in situations designed to emphasize and develop these connections. Lack of linkages limits the growth of number sense, geometric knowledge, and problem-solving ability. Studying more geometry may ameliorate this situation. The Turtle Math tasks provide experiences in which there is a mutually beneficial synthesis of the schemes of turtle-based paths, geometric properties, and arithmetic. Our studies show that those students who connected ideas had more powerful and flexible solution strategies for solving spatial problems.

Third, descriptions of students' learning should be specific. The van Hiele levels may provide a broad framework, but they are inadequate alone (see chapter 6 and

cf. Lehrer et al., 1998). For example, students struggle to learn specific length concepts and skills. At the first and third levels of solving length problems, students in our studies, did not use hash marks to segment or partition line segments. Teachers should take special care to observe how students working at these levels interpret the task, because they need to engage in activities of different types. At the third level, students may be challenged with more complex missing lengths tasks. In addition, students at the first level need to engage in partitioning and iterating lengths, continually tying the results of that activity to their counting schemes. Tasks in which applying only numerical schemes is ineffective may be especially useful. In the domain of turns and angles, students need to maintain a record of mental images of both the initial heading and final heading of an object, using a frame of reference to fix these headings (probably an externalized or internalized vertical and horizontal framework). They have to re-present the motor activity of rotating the object from the former heading to the latter, and compare that re-presented image to one or more iterations of an internalized image of a unit of turn—units of 90° or 30° for our students—or partition the re-presented turn into these units.

Other implications for curriculum design take the form of caveats for the development of computer activities and tools. The fourth implication is that any computer environment may focus on certain mathematical aspects to the neglect of others. The Turtle Math activities are no exception. The presence and measure of turns/angles, the measures of sides, and so on, are properties that students abstract from their Turtle Math work and apply in other situations. However, implicit properties of the turtle-generated shapes, such as straight (vs. curved) lines may not be as readily internalized (recall that in the *Logo Geometry* environment, the teacher used Logo to bring such features to explicit awareness). That is, the turtle always drew straight lines; students did not have to explicitly decide which command to use to make it do so.

Fifth, specific, seemingly immaterial, details of activities actually can be essential. Sometimes, we were too cavalier in making certain changes that we thought would "simplify" a screen or activity. For example, an early navigational activity of Geo-Logo included an on-screen "battery" whose energy decreased with every command the student issued. The battery motivated the students to examine code, connect the commands to the turtle's path, and combine commands and, therefore, lengths. In a later field test, the battery was omitted for simplicity's sake. This omission was a mistake. Although the students in the study eventually achieved the goal behaviors, the process was slower and less complete. As another example, the coordinate activity "Sunken Ships," which was changed from an earlier version, should not have had the distance clues based on taxicab geometry removed from it. The newer version of this activity, which merely used "near and far" clues, caused the students to guess rather than apply mathematical analysis.

Sixth, activities must be carefully designed to encourage mathematical analysis. Observations confirm previous findings that many students of this age initially rely on visual cues only instead of such analytical work as using dynamic imagery and

arithmetic to find precise mathematical and programming relations within the geometry of the figure (Clements & Battista, 1992b; Hillel & Kieran, 1988). There is little reason for students to abandon limited surface-level visual approaches unless they are challenged with tasks in which resolution requires an analytical approach.

Seventh, the final implication, regarding curriculum design, maintains that philosophic consistency is more than an academic concern. Constructivism, the driving theory behind the design principles, the software, and NCTM's *Standards*, was also an important part of the teaching philosophy of the more successful teachers. In this study, some teachers held a certain philosophy about computer use. They wanted the students to "master" a concept fully before the latter could use the computer and then required these students to complete shorter tasks that could be quickly checked for correctness. This philosophy caused the students to have less time on the computer and gave them less opportunity to creatively explore mathematics. Therefore, a consistent philosophy in pedagogy and the many components of curriculum materials is critical for reform.

In conclusion, the use of previous research on Logo (including *LG*) in deriving design principles and then the installation of the design principles in Turtle Math proved to be a positive effort in helping students exhibit desirable mathematical behaviors. However, curriculum and software design is an iterative process. Further insight into the students' learning of specific concepts could and should have direct implications on yet newer versions of Logo.

Chapter 6

Implications for Theory

We based our research on a synthesis of Piaget's and the van Hieles' theories. We made modifications to the van Hiele theory in two publications, suggesting a level lower than Level 1 in one (Clements & Battista, 1992b) and postulating substantive alterations to common interpretations of the van Hiele theory in the other (Clements, 1992). In this postscript, we summarize and update these modifications and demonstrate how findings from *Logo Geometry* support them.

According to the theory of Pierre and Dina van Hiele, students progress through levels of thought in geometry (van Hiele, 1959, 1986; van Hiele-Geldof, 1984). Thinking develops from a level driven by visual patterns through increasingly sophisticated levels of description, analysis, abstraction, and proof (Clements & Battista, 1992b). We suggest that it may not be viable to conceptualize a purely visual level (Level 1 in the van Hiele hierarchy), followed and replaced by a purely verbal descriptive level (Level 2) of geometric thinking (and so on), which is a perspective not infrequently taken in some discussions of the van Hiele theory (note that van Hiele and we have never really characterized Level 2 as purely *verbal;* "descriptive" can certainly include visual aspects). Instead, we suggest that different types of reasoning—those characterizing different levels—can coexist in an individual and develop simultaneously but at different rates and along different paths. Each path leads to slightly different combinations of multiple types of knowledge. This view implies a different conceptualization of levels of geometric thinking and of students' development of these levels.

MULTIPLE PATHS TO MULTIPLE TYPES OF KNOWLEDGE

From the perspective that we suggest, levels do not consist of unadulterated knowledge of one type only. This view is consistent with recent literature from Piagetian and cognitive traditions (e.g., Minsky, 1986; Siegler, 1996; Snyder & Feldman, 1984) as well as with reinterpretations of the van Hiele theory, including Clements (1992), which we are elaborating here, and others (Gutiérrez, Jaime, & Fortuny, 1991; Lehrer et al., 1998; Pegg & Davey, 1998). These interpretations reject the assumption that one level of geometric knowledge, such as visual knowledge, exists in an individual. Instead, for example, children at both Levels 1 and 2 possess triangle schemes that include visual/imagistic and nonvisual or verbal declarative knowledge ("knowing what") about shapes. Children who were described as "geometry deprived"[1] (Fuys et al., 1988) have a number of imagistic prototypes for triangles, such as an equilateral triangle and a right triangle, both with a horizontal base (Hershkowitz et al., 1990; Vinner & Hershkowitz, 1980).

These prototypes are not absolutely rigid, but they have constraints. For example, the more the lengths of the legs of a right triangle differ, the less likely that triangle will be assimilated to that prototype. Such prototypes can be thought of as having multivariate distributions of possible values (e.g., for the relationships between the side lengths and for the degree of the base's rotation from the horizontal) in which, the nearer to the mode of the distribution the perceived figure is (e.g., equal for the side lengths and 0° rotation of the base), the more likely it will be assimilated to that prototype (parallel distributed processing, or PDP, networks model this type of system, see Clements & Battista, 1992b; McClelland, Rumelhart, & the PDP Research Group, 1986).

Research has substantiated that children possess several different prototypes for figures (e.g., a vertically and a horizontally-oriented rectangle) without accepting the "middle" case (e.g., an obliquely-oriented rectangle). In one study, subjects studied a preponderance of rectangles with extreme values and few intermediate values of variables such as size (Neumann, 1977). Subjects were presented with test items and asked to rate their confidence that they had already studied that particular item. Interestingly, the subjects rated the extremes (e.g., large or small rectangles) much higher than items created by using the mean of these values (e.g., middle-sized rectangles), showing that they extracted multiple foci of centrality, thereby creating several visual prototypes. Thus, they did not cognitively "average" what they had studied. This finding is consistent with studies on the van Hiele theory (Burger & Shaughnessy, 1986; Clements & Battista, 1992b; Fuys et al., 1988), as are the data presented in this monograph (e.g., recall students' performance on the shape identification tasks).

Again, let us consider "geometry deprived" children's knowledge. We suggested that they have a number of constrained imagistic prototypes for triangles. They also possess verbal declarative knowledge that may include the name *triangle* and a few statements of components, such as *three sides* and possibly *three corners*. However, these statements are not constrained further; for example, there are few limitations placed on the nature of these sides (e.g., they might be curved) and corners.

Consistent with our suggestion that children possess multiple types of geometric knowledge, we suggest that the children's knowledge of geometry might be enhanced in different ways. First, their imagistic prototypes might be vastly elaborated by the presentation of a variety of exemplars, through the systematic variation of irrelevant and relevant attributes (e.g., through dynamic media such as the

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[13] Fuys et al. use this phrase to describe children with extremely limited and rigid school experience with geometry. Of course, all students receive substantial daily experience with spatial forms, and their ideas about geometric objects, such as angles, are strongly influenced by this experience and the common language patterns used to communicate about it (Clements & Battista, 1989). Although these in-school and out-of-school, socially constituted experiences are critical to ecological validity (Bronfenbenner, 1989), their full consideration lies outside the scope of this chapter. That is, as an elaboration of extant work on the van Hiele theory, this chapter shares the focus of that work on students' knowledge; however, this does not diminish the criticality of the instantiation of that knowledge in different contexts.

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computer). Children must actively attend to the exemplars and connect them through verbal labeling. Such enhancement is accomplished mainly through a usually unconscious visual induction process.

Second, through the presentation of particular tasks and engagement in dialogue about them, students' verbal declarative knowledge might be refined to extend, elaborate, and constrain their visual knowledge. The *LG* research, for example, indicates that certain experiences with the Logo computer language extend students' conceptions of the "three sides" of a triangle to include the constraints that these sides must be straight and that they all must interconnect (i.e., that a triangle must be a closed figure). Students, even those clearly not above Level 1 thinking, could use Logo terms to discuss these constraints and apply them in classification tasks, thereby rejecting shapes with curved sides and the wrong number of sides—even those shapes that share visual characteristics with triangles (e.g., a chevron). An alternative interpretation of these experiences might be that the Logo activities merely helped children construct more varied visual prototypes for triangles. However, evidence does not support this, because although the Logo students were not more accurate in identifying triangles, they did accurately identify nontriangles that had visual aspects of triangularity.[14] To accomplish such correct identification in the instance of closure, elaborated conceptions must mitigate the effects of gestalt visual tendencies, giving precedence to declarative criteria in figure classification tasks. For example, more children in our Logo classrooms, compared to the controls, rejected figures created by omitting a small segment of a prototypical triangle ( ), justifying their decision with ideas such as the following statement: "It looks like a triangle, but it can't be because triangles have to be closed all the way." This approach requires more conscious reflection, in which abstraction, induction, generalization, and short deductive chains of reasoning are combined.

We hypothesize that each of these two paths toward enhancing knowledge of geometry can be followed separately or together. If separately, knowledge of one type can "substitute" for knowledge of another type on certain tasks, within certain limits (including a performance decrement, in accuracy or speed of execution). For example, a rich and varied exposure to various exemplars could allow near-perfect performance on triangle discrimination tasks such as those discussed previously. Performance would suffer only if a figure were presented that fell outside the range of any of the multiple imagistic prototypes developed by the children or if the task demanded reasoning based on properties not supported by such imagistic-oriented schemes (e.g., calculations regarding angle relationships), in which case the schemes would be inadequate to the task.

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[14] Not all types of Logo experience necessarily build greater verbal declarative knowledge; some types of Logo experience might simply help students refine their visual images. Various types of experience may include reflection on turtle movements or involvement in class discussions about turtle movements and shapes. The former might refine students' visual imagery; the latter might build up relevant verbal declarative knowledge. In the *LG* research, these two effects are confounded.

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Similarly, well-developed verbal declarative knowledge, even in the absence of exposure to various exemplars could allow perfect performance on such simple discrimination tasks. However, performance would suffer (at least in speed of execution) because a chain of reasoning would be required for every figure that could not be immediately assimilated into the (hypothesized-limited) range of imagistic prototypes. In each of these instances, the characteristics of the figures would have to be "read off" the figure and compared to those that are held in verbal declarative knowledge as characteristics of the class. This type of verbal knowledge would support more sophisticated analysis of geometric figures. Without the imagistic knowledge, however, the range and flexibility of the application of this knowledge would be limited.

Research on the learning of geometry shows that students limit their conceptualizations to studied exemplars and often consider common but nondefining features (e.g., an altitude of a triangle is always located inside the triangle) as essential features of the concept (Burger & Shaughnessy, 1986; Fisher, 1978; Fuys et al., 1988; Kabanova-Meller, 1970; Zykova, 1969). Further, students, who know a correct verbal definition of a concept but also have a limited visual prototype associated with it, may have difficulty applying the verbal description correctly (Clements & Battista, 1989; Hershkowitz et al., 1990; Vinner & Hershkowitz, 1980). This is consistent with the theory of simultaneous and sequential processing. In this view, pictorially-presented materials may be more likely to evoke the visual/holistic/simultaneous processing, whereas aurally-presented materials would be more likely to evoke verbal/sequential/linear processing. The type of presentation may interact with individual differences in abilities in these two domains.

These hypotheses have two implications. First, at least three paths to the elaboration of geometric knowledge are possible: enhancement of imagistic knowledge, enhancement of verbal declarative knowledge, or enhancement of both (including interrelating the two). All three would lead to increased performance on low-level geometry items found on most mathematics achievement tests. Similarly, all three may lead to performance on certain tasks that could be interpreted as indicating growth toward higher levels of thinking. Enhancing only imagistic knowledge, however, would not permit sophisticated verbal conceptual reasoning. Enhancing only verbal declarative knowledge would allow such reasoning (albeit not cause or ensure it) but would place constraints on its application and generalizability (thus making it unlikely to be applied generally). Only a simultaneous and integrated enhancement of both types of knowledge would encourage the development of rich and robust geometric knowledge—knowledge that not only would support future achievement in geometry but also would allow the application of geometric knowledge in a wide range of mathematical and nonmathematical domains.

VAN HIELE MODEL OF LEVELS OF GEOMETRIC THINKING

These hypotheses also have implications for the van Hiele model of levels of geometric thinking. Conceptualizing geometric growth as being strictly visual, then

strictly descriptive/analytic, then logical, and so on, is neither completely accurate nor optimal for educational theory and practice. (Nor does strict linearity necessarily reflect van Hiele's view, though his theory is often discussed in this form.) A synergistic approach has more validity. In such a view, verbal declarative knowledge is a component of thinking at Level 1 (and, to a more limited extent, at Level 0, as will be addressed in the following paragraph), and enhancing verbal declarative knowledge solidifies geometric achievement at these levels (van Hiele, 1986, p. 83–86, allows for such a role). There is a range of such verbal knowledge components. As previously described, these include simple statements, such as "has three sides," and ostensibly palpable and simple symbols, such as the word "triangle." More than merely a label, a term such as *triangle* serves as a nexus about which images and verbal declarative ideas coalesce. In this sense, we agree with Vygotsky (1934/1986), who said,

> The primordial word by no means could be reduced to a mere sign of the concept. Such a word is rather a picture, image, mental sketch of the concept. It is a work of art indeed. That is why such a word has a "complex" character and may denote a number of objects belonging to one complex" (p. 133).

Thus, a word has several zones of unequal stability; in the context of geometric forms, these zones include the hypothesized prototypes determined by multivariate distributions. The zones are tied to certain contexts, often so tightly that they connect automatically. For example, a student who had worked with Logo turtle geometry in first grade was asked about the number of angles in a triangle in his third-grade year. He queried, "What do you mean 'angle,' ... corners?" (Clements, 1987). Later, he was asked how he had drawn a triangle with the turtle two years previously. He replied,

> We started and then I put left and I made it turn so it would be like that [rotating hand]. Then I made it go forward so it would go like that and then I made it turn on an angle. An *angle* [shouting and laughing]! A turn! A turn ... the same thing!"

This notion of multiple types of often compartmentalized knowledge is consistent with other researchers who find that Level 1 involves multiple mental operations (Lehrer et al., 1998).

In a similar vein, imagistic knowledge should not be associated solely with Level 1 thinking. Schemes for geometric forms are abstractions from active kinesthetic and visual experiences at all levels (Fischbein, 1987; Piaget & Inhelder, 1967).

There are two implications of our synergistic approach and the data that supports it for models of geometric thinking. First, we have argued that there is evidence for a level more basic than van Hiele Level 1: Level 0, which we call pre-representational, or pre-schematic (Clements & Battista, 1992b). Research from the van Hiele perspective has identified students who fail to demonstrate thinking characteristic of Level 1—including students at the high school and college level; in these studies, categorization in a level below Level 1 was reliably distinct from categorization into Level 1; it was also stable and predictive of achievement (Mayberry, 1983; Senk, 1989; Usiskin, 1982). Further, Piagetian

research suggests that geometric objects are first constructed by the student on the perceptual plane and only later are reconstructed on the representational or imaginal plane (Piaget & Inhelder, 1967). Therefore, at Level 0, children may have constructed many geometric objects at the perceptual plane. They often can match and identify many shapes (most reliably, prototypes). However, they attend only to a proper subset of a shape's visual characteristics and are unable to identify many common shapes. In tactile contexts, they can distinguish between figures that are curvilinear (e.g., a circle) and those that are rectilinear (e.g., a square) but they cannot distinguish among figures within those classes. Even in visual contexts, these children may not be able to construct an image of shapes or a re-presentation of the image. They are unable to rotate shapes and place them into part-whole relationships (Wheatley & Cobb, 1990). Thus, before Level 1, students lack the ability to construct and manipulate re-presentable visual images of geometric figures.

Second, at levels of thinking higher than Level 1, geometric knowledge continues to include nonverbal, imagistic components; that is, every mental geometric object includes one or more image schemes—recurrent, dynamic patterns of kinesthetic and visual actions (Johnson, 1987). Thus, imagistic knowledge is not left untransformed and merely "pushed into the background" by higher levels of thinking.[15] Imagery has a number of psychological layers, from more primitive to more sophisticated (each of which connect to a different level of geometric thinking) that play different (but always crucial) roles in thinking, depending on which layer is activated. Thus, even at the highest levels, geometric relationships are intertwined with images, though these may be abstract images.

Further, imagistic knowledge is used by some children as the basis for visually based reasoning. For example, this reasoning may even involve an early form of Level 3 relational reasoning dealing with figure classification. In the following dialogue, second-grade *LG* student, Coletta, stated that a square is a rectangle.

Teacher: Does that make sense to you?

Coletta: It wouldn't to my [4-year-old] sister, but it sort of does to me.

Teacher: How would you explain it to her?

Coletta: We have these stretchy square bathroom things. And I'd tell her to stretch it out and it would be a rectangle.

Here Coletta uses imagery to reason informally about the relationships between figures—a square is a rectangle because it could be stretched into a rectangle. She may have accepted certain "legal" transformations (e.g., those that preserve 90°

[15] Early work by van Hiele, and the work of his interpreters, espouse this view most directly. In more recent writings, van Hiele (1986, p. 42) altered his position of *valuing* higher levels more so than the visual level, though his idea of the psychological role of imagery is more obscure. Statements, such as "However, after the explication of those properties and relations by an analysis or discussion, the symbol loses the character of image, acquires a verbal content, and thus becomes more useful for operations of thinking" (p. 61) and "The image has fallen into the background" (p. 62), maintain the notion of the decreasing import of images, but others, such as "If abstract structures are developed from visual structures in which images play an important part, a portion of the rich (and unexpressed) contents will probably be structured too" (p. 67), suggest more agreement with the basic ideas expressed here.

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angles, which she previously had associated with both rectangles and squares). This illustrates a process of reciprocal shaping occurring between her imagery and her verbal declarative knowledge (for other examples from *LG*, recall Jonathan's and Jeremy's use of imagistic processes detailed in chapter 3). In summary, imagistic knowledge strengthens geometric thinking by supporting more robust and generally applicable reasoning. This pertains to every level of thought; for example, high school students were reported to use both Level 1, visually based reasoning, and Level 3, analytical reasoning, simultaneously in locus tasks (Hershkowitz & Dreyfus, 1991).

To repeat, our argument does not mean that the images do not change with development. The essence of Level 2 thinking, for example, lies in the integration and synthesis of properties of shapes, not merely in the recognition of shapes. At this level, children have transcended the perceptual and have constructed the properties to be singular mental geometric objects that can be acted on and not merely descriptions of visual perceptions or images (cf. Steffe & Cobb, 1988). Ideally, however, these objects are not solely "words or pictures" (Davis, 1984) but a synthesis of verbal declarative and rich imagistic knowledge, each interacting with and supporting the other. Thus, the question should not be whether geometric thinking is visual or not visual. Rather, the question is whether imagery is limited to unanalyzed, global visual patterns or if it includes flexible, dynamic, abstract, manipulable imagistic knowledge (Clements, Swaminathan, Hannibal, & Sarama, 1999). This latter type of knowledge, and the concurrent development of active and reflective visualization that acts on figures, in addition to drawings, is a viable goal at all levels of thinking. Therefore, we believe that Level 1 should be called the *syncretic level*, rather than the visual level, signifying a global combination without analysis (e.g., analysis of the properties of figures).

In transition between Level 1 and Level 2, students learn to recognize, describe, and manipulate not only individual shapes but also their components and, eventually, their properties. Note that students recognize and describe the properties of individual figures without necessarily ascribing those properties to all of the shapes in that class. Knowledge of properties of isolated shapes ("this rectangle") or classes of shapes (rectangles) eventually coalesces into knowledge of properties of two-dimensional shapes and a general disposition to make sense of novel two-dimensional situations by analyzing properties. Figures 32 and 33 illustrate the two contrasting conceptualizations of geometric levels of thinking. They encapsulate arguments to this point and set the stage for arguments developed in the following section.

NATURE OF THE LEVELS

What is to be said about the nature, especially the hierarchical nature, of the levels? First, they are indeed *levels*, rather than *stages*. Karmiloff-Smith (1984) defined a *stage* as a substantive period of time characterized by cognition across a variety of domains qualitatively different from that of both the preceding and

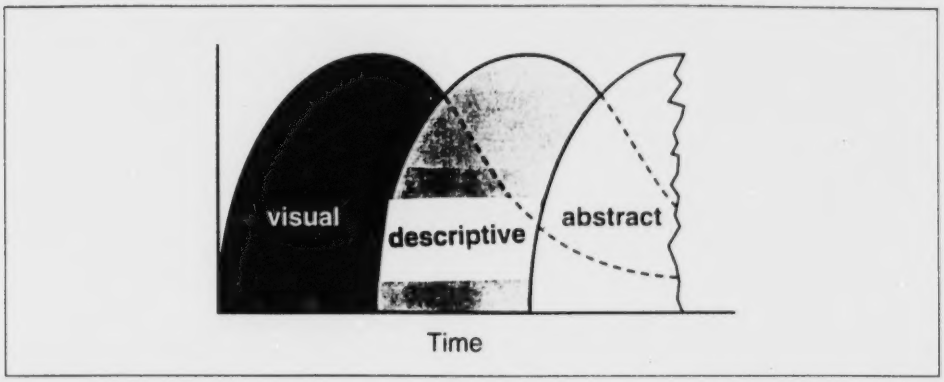

Figure 32. Hypothesized linear view of the levels of geometric thinking.

Note: In this view, each level ripens to fruition, engendering the beginning of the next level—which incorporates and subordinates the earlier level—and finally fades away.

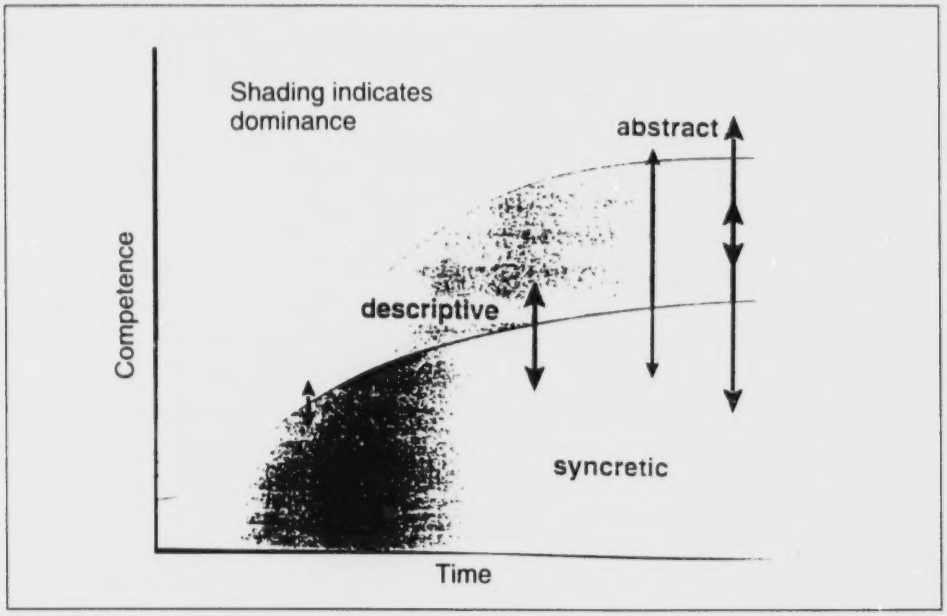

Figure 33. Hypothesized synthetic view of the levels of geometric thinking.

Note: In the synergistic view espoused here, types of knowledge develop simultaneously. Although syncretic knowledge is dominant in the early years (darker shading indicates dominance of a particular level of thinking), descriptive knowledge is present and interacts with it, though weakly (symbolized by the small double arrow at the left). Syncretic knowledge, descriptive verbal knowledge, and to a lesser extent at first, abstract symbolic knowledge grow simultaneously, as do their connections. When abstract knowledge begins ascendance, connections among all types are established and strengthened (indicated by thicker arrows). The unconscious probabilities of instantiation associated with each level are indicated only by the intensity of shading. Not shown but similar in progression are the executive processes that also develop over time. The executive processes serve to integrate these types of reasoning and, more important, to determine the level of reasoning that will be applied to a particular situation or task. When only this dominance is represented in a graph (see Figure 34), we see a version of the "overlapping waves" metaphor (Siegler, 1996). (Note that Figures 32 and 33 and the accompanying analysis were originally published in Clements, 1992.)

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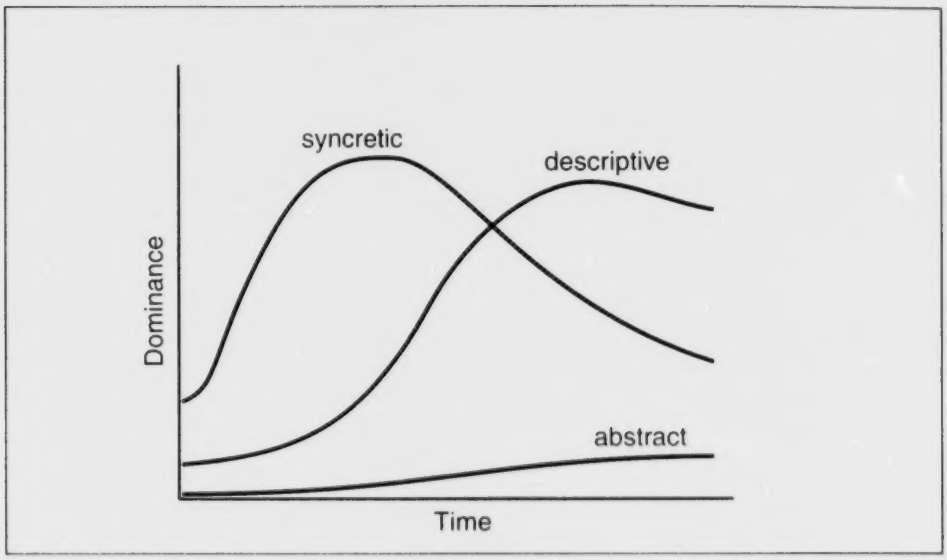

Figure 34.

succeeding stages. Similarly, a *level* is a period of time of qualitatively distinct cognition, but within a specific domain.[16] Thus, students are *at* Level 1 when their overall cognitive organization and processing disposes them to think about geometric shapes primarily in terms of visual wholes; they are *at* Level 2 when their overall cognitive organization disposes them to think about shapes in terms of their properties and when the students have the ability to do so for the shapes with which they have worked. The following two episodes illustrate the difference between Levels 1 and 2 with students working in *LG*.

A second grader was examining her attempt to draw a tilted square in a version of Logo that allowed students to erase commands and their graphic effects. This student used a trial-and-error approach, giving one command at a time, erasing a command if the part of the figure the command drew did not look right. Because she did not use 90° turns, the figure this students produced was only a crude approximation of a square. But she concluded that it was a square, reasoning as follows.

Interviewer: How do you know it's a square for sure?

Michelle: It's in a tilt. But it's a square because if you turned it this way, it would be a square.

Michelle did not refer to properties; her decision was based on her thinking that the figure will look like a square when turned. Contrast this visual response to that

[16] There are other interpretations of levels. For instance, they are horizontal markers on a vertical scale of measurement with no temporal component (Glaserfeld & Kelley, 1982). We use Karmiloff-Smith's interpretation here.

of two fifth graders who had drawn a tilted square by using equal forward commands and 90° turns.

Interviewer: Is it a square?
Sammy: Yes, a sideways square.
Interviewer: How do you know?
Sammy: It has equal edges and equal turns.

Though these students were dealing with the same task in similar environments, the basis for their decision making and figure identification was different. The second grader constructed her figure visually and used visual means to identify it. Her justification was disconnected from the process she used to construct the figure but was still visual. In fact, she had stated earlier that squares have all sides equal. But, although she tried to implement this equality in her Logo procedure by making the inputs to the forward commands all 30, she abandoned it by adding two forward 5 commands for the last side, in order for the figure to close. The fifth graders, on the other hand, knew the properties of a square and knew how to use Logo to imbue their figure with those properties. They justified their identification by referring to those properties.

An additional insight can be gleaned from the first episode. According to van Hiele, for students at the visual level, "There is no why, one just sees it" (van Hiele 1986, p. 83). He continues by saying,

> Without the existence of a network of relations, reasoning is impossible. The first speaker [a student at Level 1] did not arrive at his judgment by means of reasoning. He saw the rhomb and that was sufficient. Nor would it be possible to have his judgment withdrawn by means of reasoning (p. 110).

As Michelle's remarks indicate, however, there was a "why" for her. She reasoned that the figure in front of her was a square by performing a visual transformation on it and by comparing the transformed image to her image of a square. It is not as simple as van Hiele's description might suggest; Michelle did not simply say "It's a square because it looks like a square." Some students at the visual level may utilize only simple visual comparisons to make judgments. But this episode shows us that others, who are perhaps more advanced but still at the visual level, use more sophisticated visual thinking. Note, however, the limitations of her imagery. She visually concludes that the figure she drew could be turned to look like a square in standard orientation. She does this even though the figure drawn does not have all the properties that she previously named for squares (i.e., that all the sides are equal). Therefore, her visual imagery was not strongly connected to her verbal, property-based knowledge.

Neither stages nor levels are recurrent (Karmiloff-Smith, 1984); that is, a student at stage (level) n does not return to stage (or level) $n - 1$. Van Hiele (1987) claims that the levels of thinking apply to both geometry and other mathematical topics such as arithmetic, but he also claims that children usually are at different levels within these different topics.

Further, most empirical studies suggest that on different subtopics of geometry, people exhibit behaviors that are indicative of different levels (Denis, 1987; Mason,

1989; Mayberry, 1983) and sometimes favor reasoning at different levels on different tasks (Burger & Shaughnessy, 1986; Clements & Battista, 1992b). Assessment assigning people to a single level forces a partial loss of the richness of students' responses. In several *LG* interview tasks, codes such as "1.5" were used to more accurately indicate students' responses (cf. Gutiérrez, Jaime, Shaughnessy, & Burger, 1991). The finding of distributed responses across levels suggests that the domain at which levels of thinking operate may be, at most, subtopics within the broad topic of geometry, although a level reached in one subtopic may catalyze learning to think at that level in a different subtopic. (Recall our elaborations regarding levels of thinking for polygons and motions in chapter 3.)

Summarizing, there are three reasons to consider reconceptualizing the nature of levels of geometric thinking. First, the number of levels (and even the mapping between these different interpretations) is ambiguous. Second, there are reports of stronger and weaker performance at certain levels. Third, people's performance is reported to be spread across levels. Thus, the question emerges: "How wide a band can be permitted before the present notion of hierarchical dependency must be reformulated?" (Clements & Battista, 1992b). Further, there are few signs of discontinuities between levels that are postulated as fundamental to the theory (van Hiele, 1986, p. 49). Also, growth in small increments is observed more frequently than large jumps in learning (Burger & Shaughnessy, 1986; Fuys et al., 1988; Lunkenbein, 1983; Usiskin, 1982). Consistent with these results, we take the position that learning is incremental (see Figure 33) rather than intermittent and tumultuous (i.e., occurring mainly between stable levels).

Also, it might be argued that the levels hierarchy is more *logical* than *psychological* in nature (Clements & Battista, 1992b). There are two theoretical rationales for this argument. The first argues that there may be no other way that behaviors at each level could possibly be sequenced on logical grounds; if that is so, then the hypothesized sequence is not explanatory (or even particularly descriptive) in any psychologically meaningful way and does not need empirical verification. The second argues that the sequence may be an artifact of external and arbitrary educational engineering. We discuss each of these rationales, in turn, in the following paragraphs.

First, the set of levels might constitute what Brainerd (1978) has called a "measurement sequence." He postulates that often a behavior sequence is not "in the organism" but rather "in the tests"; that is, the sequence results from definitional connections between the behaviors being measured. A measurement sequence occurs when tasks, measuring behaviors at each stage or level, require the knowledge and procedures of the preceding levels as well as additional knowledge or procedures. Thus, invariant sequences must, by logical necessity, be "found" by empirical studies because it is not possible to devise valid tests that do not measure earlier items. This is undoubtedly a valid concern in some instances. However, it is not certain that we can a priori and logically determine that a task requires the knowledge and procedures of the preceding levels. In some cases, for example, learning simple number concepts like counting and cardinality before learning to

multiply, such a determination might be made. However, Brainerd gives as an example "learning to add before learning how to multiply" (p. 177); in this instance, it is possible that the a priori prerequisite status, "logical" to most adults, would not apply to every student in every situation. The mathematics education research literature indicates that some "logical prerequisites" do not always precede those abilities for which they are purportedly prerequisite (e.g., Hiebert, 1981).

The applicability of this criticism may vary, depending on the contiguous stages one considers. For example, logically ordering *properties* (Level 3) or constructing proofs dealing with those properties (Level 4) would seem to require awareness of those properties (Level 2) by logical necessity. In this instance, Brainerd's aforementioned measurement sequence criticism might be applicable. However, it is not clear if logically ordering classes of figures requires syncretic (Level 1) knowledge in the same way. This leads to the second theoretical criticism.

The second theoretical rationale is that the levels might result from *curriculum* sequences. That is, the empirical results, that level n appears before level $n + 1$, may be an artifact of the curriculum sequence experienced. There is some support for this notion. For example, even young children can be instructed in a nonvisual, descriptive manner, leading to higher achievement on certain concepts. In particular, relationships for geometric figures that do not have a small number of visual prototypes (templates) and do involve relatively simple deductive argument, such as quadrilaterals, can be taught to first graders (Kay, 1987). In a similar vein, de Villiers (1987) reported that hierarchical class inclusion and deductive thinking could develop independently in eighth-grade and ninth-grade students, and that hierarchical thinking depends more on teaching strategy than on van Hiele level. Resequencing the curriculum and therefore, ostensibly, students' learning of geometric ideas appear to weaken the criticism that the levels constitute measurement, rather than development, sequences. However, by the same token, such resequencing, raises the possibility that the order of emergence of the types of thinking at each level is neither logically (measurement) *nor* developmentally constrained.

Despite empirical findings that appear troubling to the theory and despite the arguments that the levels may be the result of measurement or curriculum sequences rather than psychological sequences, we believe it is too early to demote the levels of geometric thinking to a framework made up only of nonordered types of thinking. Other evidence on the hierarchical ordering is fairly consistent (Denis, 1987; Gutiérrez & Jaime, 1988; Mayberry, 1983), although there are exceptions (Mason, 1989). In addition, the *LG* data indicate that, presented with appropriate geometric tasks and problem-solving tools (in this case, modified Logo environments), young children, who initially respond at the syncretic level, construct ideas about properties of geometric figures and come to appreciate the power of using this knowledge in the solution of problems.

We hypothesize that, even though all types of thinking do grow in tandem to a degree, a critical mass of ideas from each level must be constructed before thinking characteristic of the subsequent level becomes ascendant in the student's orientation toward geometric problems (see Figure 33). Thus, while multiple levels

develop incrementally and simultaneously, one level maintains a privileged status (Minsky, 1986, describes a similar hypothesis about stages of development). This status is achieved by both intentions and interschematic relationships that yield an internally-computed probability of successful response for each level of thinking that is applied to the given situation. Successful application leads to the increasing use of a particular level. However, under conditions of increased task complexity, stress, or failure, this probability level decreases and an earlier level serves as a fallback position (Hershkowitz & Dreyfus, 1991; Siegler, 1986).

No level of thinking is deleted from memory (Davis, 1984). That is, re-recording a mental representation at a more explicit level does not erase the earlier representation. (Indeed, replacing or deleting any successful representations would eliminate "fallback" strategies that arguably are essential when new, untested knowledge is being formed, cf. Minsky, 1986.) The schemes that constitute the representation are instantiated dynamically (i.e., their form is interactively shaped by the situational context, including the task demands); and they can be modified at this time (or new schemes can be formed), although schemes that are basic subschemes of many other schemes become increasingly resistant to alteration because of the disruptive effect alteration would have on cognitive functioning. The syncretic (visual) level has a special status in this regard. Due to daily life in a spatial world, this level receives constant application. Consequently, it is maintained as a separate system, and the probability of its instantiation (construction of a mental model of the world in a particular situation) also is consistently maintained. Therefore, this level is an interpretive and default behavior for most people in most situations that they interpret to be nonmathematical. When the situation requires an interpretation from a mathematical perspective, however, a schematic shift is likely to occur that leads to a rapid interpretation (or reinterpretation) at the ascendant level of the person involved.

This situation poses a problem for postulating levels of geometric thinking. That is, levels are theoretically nonrecurrent (Karmiloff-Smith, 1984); however, people not only can, but frequently do, "return" to earlier levels of geometric thinking in certain contexts. In the *LG* research, this is illustrated most clearly in the triad task. (See also Lehrer (1998); Pegg (1998) provides different examples.) Therefore, we postulate the construct of *nongenetic levels*.

Nongenetic levels have two special characteristics. First, progress through nongenetic levels is determined more by social influences, and specifically instruction, than by age-linked development. (At this point, this only implies that progression does not occur by necessity with time but demands, in addition, instructional intervention. Certain levels may develop under maturational constraints; further research is needed on this issue.) Second, although each higher nongenetic level builds on the knowledge that constitutes lower levels, its nongenetic nature does not preclude the instantiation and application of earlier levels in certain contexts (not necessarily limited to especially demanding or stressful contexts). For each level, there exists a probability of evoking each of numerous different sets of circumstances. However, this process is codetermined by conscious metacognitive control,

and this control increases as one moves up through the levels. Therefore, people have increasing choice to override the default probabilities.

The use of different nongenetic levels is environmentally adaptive; thus, the adjective "higher" should be understood as a higher level of abstraction and generality, without the implication of either inherent superiority or the abandonment of lower levels as a consequence of the development of higher levels of thinking. Nevertheless, the levels would constitute veridical qualitative changes in behavior, especially in regard to the construction of mathematical schemes (i.e., construction of geometric objects) out of action.

PHASES OF LEARNING

Some researchers have maintained that geometric knowledge is initially learned in conscious, declarative form (Anderson, 1983). Proceduralization gradually replaces the original interpretive application with productions that perform the behavior directly. For example, after initially learning and verbally rehearsing the geometric side-angle-side congruence rule and then figuring out how it applies, students build a production that directly recognizes the application. This declarative-to-procedural sequence may accurately portray certain types of learning, especially with older students in traditional educational environments. We posit, however, that an inverse sequence better depicts much learning of geometry, especially at the lower levels. In this view, students' initial knowledge is proceduralized and schematicized at a low level to which there is no conscious access. Gradually, this knowledge is mentally re-represented so that it becomes available, as data, to other components of the cognitive system and eventually to consciousness. (The "doing to understanding" sequence also might occur in later phases of learning, based on the interaction between symbolic and visual modes of thinking, on the partial layers of discrimination that are constructed, and on the way in which the computer acts as cognitive scaffolding for the learner, cf. Hoyles & Noss, 1987a.)

This view is consistent with the work of Karmiloff-Smith (1984; 1986; 1990), who postulates a repeating three-phase cycle of representational redescription. Phases, in contrast to structurally constituted stages or levels, are recurrent, general (across-domain) processes that people work through during development or learning. At Phase 1, the building of mental representations is predominantly driven by interactions of the students' goal-directed schemes with the environment. The students' goal is behavioral success, or the reaching of that goal, which sometimes is evaluated by consistency with adult responses and feedback.[17] Such success

[17] This sentence could be misconstrued to mean that students are controlled by the external environment. We assume students are always sense-making beings; however, during this phase in building a representation, they are actively making sense of their social and physical environments, rather than their representations, of which they are not yet conscious. This active sense making is critical, in that it allows the students to differentiate between environments that do and do not assist goal attainment.

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leads to the recording of isolated correspondences between environmental situations that do or do not allow or aid the attainment of the goal in a form inaccessible to the system (i.e., input-output correspondences along with their contexts in compiled form). In this form, any relationship between bits of knowledge are, at best, implicit.

When behavioral success is achieved, a Phase 2 metaprocess (a procedure that operates on internal knowledge structures) evaluates the knowledge base. Now, the goal is not to behave successfully, but to gain control over the representational forms (Vygotsky, 1934/1986, similarly stated that the development of thought cannot be derived from the failure of thought and postulated a genetic predisposition to gain control of mental representations). The first operation of Phase 2 is to re-record Phase 1 representations in a form which can be accessed, though not yet consciously. The implicit representations are analyzed into semanticized components, linking them into a simplified but growing network structure that is predicated on the usefulness of the initial correspondences to goal attainment. The second operation is to form relationships between bits of knowledge. These two new operations place demands on cognitive processing, which together with the (over)simplified structure and the need to test the mental representation in new situations, often leads to new "errors"—ostensibly a step backward to Phase 1 from an adult's perspective—that mask the progress in explicating representations of the domain. Finally, these two operations constitute an internalization of relationships and processes that were previously only implicit (cf. Steffe & Cobb, 1988).

Once successful re-representation is achieved, students develop Phase 3 control mechanisms that integrate and balance consideration of the external environment and the new internal representational connections forged in Phase 2. At the end of Phase 3, these connections are re-recorded again in abstract symbolic form, the first form accessible to conscious thought. Now, performance improves beyond that which was achieved at either of the two earlier phases.

At the syncretic level of geometric thinking, students are implicitly recognizing the properties of shapes. For example, their schemes for squares and for rectangles both contain patterns for right angles. But these are patterns in spatial subsystems that emerge when instantiated; they are not conceptual objects (mental entities that can be manipulated or scrutinized, Davis, 1984). Right angles are not represented explicitly and, therefore, no relationship is formed between them. In general, this type of representation explains how operation at one level can presuppose knowledge from the succeeding level, without allowing access to knowledge at the higher level because the form of such knowledge is proceduralized or schematized behaviorally and is, therefore, inaccessible to the rest of the cognitive system.

When shapes are dealt with successfully on the level of behavior, metaprocesses re-record the mental representations, creating a mental geometric object for the visual image of the right angles and a link between these mental objects for different cases of right angles, including links between those in rectangles and those in squares. Because students in Phase 2 are seeking control over their representa-

tions of these geometric forms, some increases in "errors" (from an adult's point of view) may occur (e.g., maintaining the notion that a nonrectangular parallelogram is a rectangle "because you're only looking at it from the side").

Eventually, the properties become conceptual objects that are available as data to conscious processes. That is, visual features become sentient in isolation and are linked to a verbal label; and the student becomes capable of reflecting on the visual features and thus can explicitly recognize the shape's properties. At first, however, the flexibility of application is limited. Indeed, students can be expected not only to think at different levels for different topics (Clements & Battista, 1992b) but also to think at different phases for different topics. Problem solving and discussion involving the geometric objects help build connections between the constructed knowledge (e.g., of right angles) and other similarly accessible knowledge (e.g., of parallelism and side lengths of rectangles and squares). Eventually, connections are built between properties of figures such as rectangles and squares and of properties of other geometric objects. In this manner, students' geometric knowledge can become increasingly abstract, coherent, and integrated because it is freed from the constraints of compiled, and thus inflexible, mental representations. At each level, the degree of integration increases as the connections span greater numbers of geometric, and eventually, nongeometric topics (cf. Gutiérrez & Jaime, 1988). Here again, instruction has a strong influence. Ideally, it encourages unification; however, the instruction of isolated bits of knowledge at low levels retards such development. Unfortunately, the latter is pervasive in both curriculum materials (Fuys et al., 1988) and teaching (Clements & Battista, 1992b; Porter, 1989; Thomas, 1982).

To place instructional implications in a different light, it is only after the third phase that students become explicitly aware of their geometric conceptualizations; therefore, it is after Phase 3 that the last three instructional steps in the van Hiele model (explicitation, free orientation, and integration) can begin.[18] An implication is that short-circuiting this developmental sequence (e.g., by beginning with explicitation) is a serious pedagogical mistake. Deprived of the initial construction of their own mental geometric objects and relationships (images), students construct Phase 1 (behaviorally "correct") verbal responses on the basis of "rules without reason" (Skemp, 1976). A more viable goal is the construction of mathematical meaning from actions on geometric objects and subsequent reflections on those actions.

In summary, development of geometric properties as conceptual objects leads to pervasive Level 2 thinking. This development represents a reconstruction on the abstract/conscious/verbal plane of those geometric conceptualizations that Piaget and Inhelder (1967) hypothesized were first constructed on the perceptual plane

[18] The first two steps, information and guided orientation, of the van Hiele model's five-step instructional sequence would be, with some modification, consonant with the three phases of development described here. This topic is related to, but different from, the topics addressed here and will not be examined.

and then reconstructed on the representational/imaginal plane. Thus, Level 2 thinking requires what Piaget called the *construction of articulated mental imagery,* which develops most fully through the combined enhancement of both imagistic and verbal declarative knowledge, as previously discussed. We posit that the same recurrent phases explain the reconstruction on each new plane (level) but that the role of social interaction and instruction increases in importance with higher levels.

Finally, the theory of reiterated phases of re-representation applies equally to children and adults, regardless of their overall stage of cognitive development (Karmiloff-Smith, 1990). This is consonant with reports of low van Hiele levels among high school students and preservice teachers (Burger & Shaughnessy, 1986; Denis, 1987; Gutiérrez & Jaime, 1988; Mayberry, 1983; Senk, 1989).

LEARNING GEOMETRY WITH LOGO: A RETROSPECTIVE LOOK

Our theoretical analysis returns us full circle to a justification for the use of Logo or other similar computer environments for learning geometry. Drawing a geometric figure on paper, for example, is for most people a highly proceduralized and compiled process. Such a procedure is always run in its entirety. This is especially true for young children, who have not re-represented the sequential instructions that they implicitly follow. Then, they cannot alter the drawing procedure in any substantive manner (Karmiloff-Smith, 1990), much less consciously reflect on it. In creating a Logo procedure to draw the figure, however, students must analyze the visual aspects of the figure and their movements in drawing it, thus requiring them to reflect on how the components are put together. As (Papert, 1980a) states, "[When] intuition is translated into a program, it becomes more obtrusive and more accessible to reflection" (p. 145). In our words, Logo programming provides the motivation, context, and language to support the re-recording of Phase 1 proceduralized and compiled drawing processes into explicit mental representations accessible to consciousness. Challenging students to alter their paper-and-pencil drawing techniques may also be beneficial. For instance, they might be asked to draw a figure in a different size or orientation or to draw all the components of a figure but in a different order (e.g., all the corners first). However, in Logo students have to specify detailed, specific steps to an noninterpretive agent. The results of these commands, and the commands themselves, can be observed, reflected on, and corrected; Logo serves as an explicative agent. As may be recalled, an example of this was when fifth-grader Jonathan struggled to decide whether he could draw each of a group of figures with a rectangle procedure, which took the lengths of the sides as input. Both Logo and the activity helped Jonathan work through this problem. In one case, he did not even have to try his second solution. When deciding what inputs to use, he recognized that the relationship between adjacent sides themselves was consonant or not (in this instance, not) with the implicit definition of a rectangle in the Logo procedure. His first attempt using Logo and his "running through the procedure in his head" contributed to this emergent certainty. (In chapter 5, we discussed the imagistic processes Jonathan used.)

Similarly, students in a Site 1 *LG* classroom had explored the notion that a square was a rectangle—a special type of rectangle. They then had created a parallelogram with Logo. One of the students came up to his teacher the next day and said that he was thinking about parallelograms at home. "Is a rectangle a special parallelogram?" he asked. "Why do you say so?" "Because it's just like the rectangle procedure if it had 90° turns." This conversation shows that the student had used his Logo experiences to extend his thinking about relationships between polygons.

More research, however, is needed before firm conclusions can be drawn regarding our reconceptualization of levels of geometric thinking. In particular, the types of thinking that define each level must be further clarified to adequately test the theory of levels of geometric thinking against alternative theories such as measurement sequences and nonordered, curriculum-driven sequences. Further, questions need to be asked, such as the following. Why exactly does level $n + 1$ develop after level n? What are the psychological functions of thinking processes at level n that lead both to increased mastery of the environment and to the eventual construction of level $n + 1$ (i.e., in what ways are level n thinking processes necessary and beneficial, rather than merely deficient)? How well does knowledge at level n have to be constructed before knowledge at level $n + 1$ knowledge begins its ascendance, and why? What levels of thinking are mathematically useful for the student and not merely useful as analytical tools for the teacher or researchers? What internal and external events mediate sequenced growth through the levels?

Finally, movements to richer imagistic and verbal declarative knowledge constitute different, albeit mutually reinforcing, paths from Levels 0 and 1 to higher levels of geometric thinking. As was argued previously, both types of knowledge (including short deductive chains right from the beginning) and their interaction ideally should be facilitated by geometry instruction. One of the main implications of the preceding critique for educational practice is that even accepting the veracity of the levels in one form or another, teachers should expect and encourage use of different types of thinking—visual, descriptive, and logical—at every level, albeit in different ratios and with different force. For example, early logical thinking may consist of short deductive chains that are instantiated only as concrete statements rather than as abstract, generalized statements.

To summarize, Logo programming can help students construct elaborate knowledge networks (rather than mechanical chains of rules and terms) for geometric topics. There are several unique characteristics of Logo that can facilitate students' learning, which have been developed in this monograph or in previous reviews and discussions (Clements & Meredith, 1993; Clements & Sarama, 1997; Noss & Hoyles, 1992). Note that claims or conjectures have not been included here. Instead, the following list constitutes hypotheses that have received empirical support.

The first set of characteristics involve the Logo environment, per se. The nature of various Logo environments, especially those enhanced to aid mathematical learning, provides certain educational advantages.

- The commands and structure of the Logo language can be consistent with mathematical symbols and structures in ways that are pedagogically useful. For

example, turtle geometry commands such as FD and RT focus students' attention on critical aspects of figures.

- Drawing with Logo's turtle graphics and creating runnable code are meaningful and interesting tasks for students; these tasks then motivate the use and learning of geometric and other mathematical ideas (Ainley, 1997).
- The turtle world involves measures that are visible, quantifiable, and formalizable, helping to connect spatial and numeric thinking (Clements et al., 1996; Clements et al., 1997; Noss & Hoyles, 1992).
- Logo can encourage the manipulation of screen objects in ways that facilitate students' thinking of them as mathematical objects and thus as representatives of a class. In this way, Logo can evoke more abstract geometric thinking.
- Logo can promote the connection of symbolic with visual representations, therefore, supporting the construction of mathematical strategies and ideas from initial intuitions and visual approaches (Clements & Sarama, 1995; Noss & Hoyles, 1992). Thus, it can serve as a transitional device from physical movements to more abstract mathematical conceptualizations. It can make mathematics more concrete while simultaneously supporting the students' formalization of actions algebraically as a computer program (Hoyles, 1993).
- Logo can permit students to "outline" or "sketch" their ideas or problem-solving plans and then elaborate and correct them. Students can work on specific cases as they work "top down" (Hoyles & Noss, 1987a) or as they think about generalizations. Interaction with the computer can direct attention to key points in the mathematical problem solving.
- Students can use Logo's symbolic code to reduce the cognitive load of a task, dealing with one relationship at a time and then synthesizing them all into a final procedure (Hoyles & Noss, 1987a).
- Logo can be a medium for the expression of mathematical ideas (Hoyles, Healy, & Pozzi, 1992). It can provide tools and motivation to make relationships explicit. Especially with appropriate tools and structure (Clements & Sarama, 1995), it can reduce the "cost" of changing the expression of ideas or experimenting with parameters (Kynigos, 1993).
- This externalization of ideas can also serve as media for communication of mathematical relationships (Hoyles, 1993).
- Students' thoughts are captured in their commands and can be reenacted and revised. Their actions, and the turtle's actions that they command, can become objects of mathematization and reflection (Clements & Sarama, 1997; Kieren, 1992; Noss & Hoyles, 1992). Students can also return to test out ideas. That is, they can always return to "run" their code and thus return to action-oriented ways of thinking as they extend higher order ideas (Kieren, 1992).
- Logo can help document student actions, leading to meaningful mathematical symbolization. Students can build a symbolic mathematical language that is based on this Logo documentation.

- Logo can provide scaffolding for mathematical analysis; that is, a symbolic representation on the computer can allow the student to erect scaffolding around the solution of a problem, and subsequently attend to only those elements of the solution for which details need to be filled in (Noss & Hoyles, 1992).
- Logo provides feedback. Logo's feedback is distinct from "right or wrong" feedback associated with CAI materials; instead, it is an instantiation of students' expression of geometric ideas. With extended environments, such as Turtle Math, students receive symbolic feedback when they manipulate a graphics object and graphic feedback when they edit their symbolic code (Sarama, 1995).
- Logo can require and facilitate precision and exactness in mathematical thinking.
- Logo can provide a window to students' mathematical thinking (Noss, Healy, & Hoyles, 1997; Weir, 1987). Such an environment provides a fruitful setting for teachers willing to work with, and listen to, students.
- Logo can help students construct more viable knowledge because they are constantly evaluating a graphical manifestation of their thinking. Further, students more ably apply mathematical knowledge in problem-solving situations.
- Because students may test ideas for themselves on the computer, Logo environments aid students in moving from naive to empirical to logical thinking. These environments also encourage students to make and test conjectures. Thus, Logo facilitates students' development of autonomy in learning (as opposed to seeking authoritative opinion) and positive beliefs about the creation of mathematical ideas.
- Logo can help structure students' play to encourage symbolic and mathematical characteristics of exploratory mathematical activity (Hoyles, 1993). Or, as Papert (1993) states, "The computer simply, but very significantly, enlarges the range of opportunities to engage as a *bricoleur* or *bricoleuse* in activities with scientific and mathematical content" (p. 145).
- Even if used for less exploratory purposes, Logo allows students to search for relationships that seem beyond their grasp at that moment; they can try a range of possibilities on the computer (Battista & Clements, 1986; Hoyles & Noss, 1989), whereas on paper they often become stuck.

Consistent across studies is the finding of the critical role played by both the curriculum, in which Logo is embedded, including the tools and structure of the version of Logo itself, and the teacher. The second set of characteristics below, is a summary of the benefits of a sound curriculum followed by suggestions that may help provide effective instruction.

- Curricula must find the right level of representation for students. For example, it may not be an efficient use of time to have students write Logo procedures to perform geometric motions, but they can use tools available in enhanced Logo environments to do so (Battista & Clements, 1988b; Clements & Sarama, 1996; du Boulay, 1986).

- Carefully planned and researched sequences of Logo activities lead to a greater likelihood of learning.
- By being intrinsically rewarding, these activities should encourage analysis.
- Versions of Logo with tools specifically designed to enhance mathematical activity lead to a greater likelihood of mathematical learning than versions without such tools (Clements & Sarama, 1995; Kynigos, 1993; Sarama, 1995).
- Extended experience with Logo may be important to realizing its maximum benefits.
- The best use of Logo may involve full integration into the mathematics curriculum. Too much school mathematics involves exercises devoid of meaning. Logo is an environment in which students use mathematics meaningfully to achieve their own purposes. The Logo language is a formal symbolization that students can invoke, manipulate, and understand (Hoyles & Noss, 1987b). Using Logo in this way can help fulfill the early vision of "teaching students to be mathematicians versus teaching about mathematics" (Papert, 1980b, p. 177).

Effective teacher mediation requires multiple actions. Teachers must be involved in planning and overseeing the Logo experiences to ensure that students reflect on and understand the mathematical concepts. Teachers might use the following approaches.

- Focus students' attention on particular aspects of their experiences.
- Educe informal language and help connect this informal language to Logo representations and to formal mathematical language for the mathematical concepts.
- Suggest paths to pursue.
- Facilitate disequilibrium by using computer feedback as a catalyst.
- Encourage mathematical analysis.
- Continually connect the ideas developed by the students to those from other contexts, such as real-world situations.

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Tables

Table 1
Number of Student Participants by Treatment, Site, and Grade

Grade	*LG* (1988–89) Site 1	*LG* (1988–89) Site 2	Control (1988–89) Site 1	Control (1988–89) Site 2	*LG* (1987–88) Site 1
K	25	37	26	34	19
1	44	45	43	47	45
2	53	52	50	44	54
3	26	24	30	23	12
4	82	48	76	44	78
5	48	48	55	47	48
6	73	51	68	57	49
7					19
Total	351	305	348	296	324

Note. Because the main comparisons for the Pre-Post paper-and-pencil test involved the *LG* and control groups participating during 1988–89, these columns are presented first in the tables within this section.

Table 2
Means and Standard Deviations for Grades K–6 Total Scores

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	48.89	70.65	54.16	59.49	52.77	55.34	57.18	60.94	43.66	67.03
	(9.30)	(6.12)	(6.13)	(15.61)	(4.86)	(6.51)	(5.34)	(6.80)	(8.14)	(6.84)
1	52.65	73.76	56.60	67.70	56.21	68.63	58.35	64.54	48.84	64.45
	(5.36)	(7.36)	(6.40)	(15.43)	(3.86)	(9.93)	(4.89)	(6.52)	(6.19)	(5.58)
2	57.29	68.81	50.44	69.02	56.53	67.93	52.70	56.80	49.29	65.36
	(9.51)	(14.99)	(13.96)	(6.68)	(5.91)	(9.04)	(12.57)	(11.77)	(6.64)	(7.98)
3	58.74	65.67	54.02	68.35	52.78	55.55	55.65	59.41	51.67	
	(5.21)	(4.43)	(7.74)	(7.16)	(6.14)	(8.37)	(9.70)	(19.64)	(3.74)	
4	61.14	74.48	60.14	65.51	59.89	70.97	62.81	60.54	55.46	68.97
	(9.00)	(8.09)	(10.88)	(22.87)	(8.97)	(14.51)	(6.13)	(17.25)	(7.95)	(6.24)
5	71.28	77.60	64.46	77.96	71.20	76.89	66.51	72.50	67.92	71.35
	(8.43)	(6.35)	(7.00)	(6.51)	(6.86)	(6.37)	(7.73)	(7.85)	(7.07)	(14.88)
6	68.96	75.00	71.67	77.96	68.62	69.66	67.74	73.91	60.53	65.89
	(9.76)	(6.90)	(6.36)	(12.89)	(9.97)	(10.77)	(9.01)	(7.15)	(7.80)	(11.02)
7									68.24	75.69
									(12.75)	(4.68)

Notes. All data except the last two columns are from the 1988–89 field test, which included matched experimental and control groups. Thus, for the first four columns, *LG* teachers at Site 1 had two years experience with the *LG* curriculum, whereas those at Site 2 had only one year experience. Data in the last two columns, from the 1987–88 field test, came from a similar test, but there were differences in items that affect the score. Missing data was mistakenly not collected by the third-grade teacher.

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Table 3
Treatment × Time × Grade × Site Interactions for 88–89: Post Hoc Analyses

	K–6 Total				4–6 Total				Triangles (Items 1.1–1.3)			
	LG		Control		*LG*		Control		*LG*		Control	
Grade	1*	2	1	2	1	2	1	2	1	2	1	2
K	+++	+							++	+		+
1	+++	+	+	+					++	++	+	
2	+	++	+						+	++	+	
3	++	++							++			
4	+		+		++	++	+	+	++	+	+	
5	+	++	+	+	++	++	+	+		++		
6	+	+		+	+++	++		+	++			+

* *Note.* 1 = Site 1. 2 = Site 2.

Key: + = Significantly improved from pretest to posttest
++ = In addition, significantly outperformed other treatment (either *LG* or control) group at that site
+++ = In addition, significantly outperformed all other groups

Table 4
Means and Standard Deviations for 4–6 Total Scores

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
4	72.78	97.50	73.23	95.06	72.01	91.80	78.79	86.96	79.07	109.02
	(12.76)	(11.87)	(10.90)	(11.64)	(10.92)	(11.40)	(8.74)	(9.62)	(12.86)	(10.61)
5	92.48	105.81	79.54	106.33	90.01	100.39	83.74	94.96	105.10	122.34
	(12.25)	(9.42)	(9.89)	(9.68)	(10.63)	(10.50)	(11.55)	(12.87)	(11.78)	(8.91)
6	87.48	100.81	93.61	108.01	86.43	90.12	87.70	99.25	91.55	106.31
	(16.15)	(11.12)	(9.43)	(9.91)	(15.45)	(16.58)	(11.07)	(10.25)	(13.61)	(12.41)
7									111.39	119.64
									(13.18)	(6.64)

Table 5
Means and Standard Deviations for Writing the Numbers of All the Figures That Are Triangles, Rectangles, or Squares (Items PP/1.1–1.3)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
					Triangles					
K	8.39	11.40	9.30	10.74	10.38	9.84	8.91	10.56	8.11	10.71
	(2.79)	(1.93)	(2.21)	(1.90)	(1.60)	(2.34)	(1.80)	(1.94)	(2.35)	(1.72)
1	8.82	11.95	9.04	11.31	9.53	10.92	9.38	9.59	8.38	10.15
	(1.82)	(1.85)	(2.16)	(1.93)	(1.64)	(2.97)	(1.91)	(1.89)	(2.10)	(1.99)
2	10.28	11.94	9.53	11.50	10.47	11.23	9.38	9.63	8.81	11.32
	(2.20)	(1.98)	(2.43)	(1.76)	(2.16)	(3.04)	(2.33)	(2.33)	1.99	(1.97)
3	10.39	11.48	9.96	9.43	9.03	9.33	10.87	10.62	10.08	–
	(1.70)	(1.62)	(1.97)	(2.04)	(1.94)	(1.68)	(1.63)	(1.80)	(1.56)	
4	11.00	12.14	10.00	12.00	10.20	11.89	10.24	11.10	9.95	10.90
	(1.98)	(2.14)	(2.30)	(1.79)	(2.16)	(2.06)	(2.07)	(1.88)	(2.11)	(2.27)
5	11.59	12.24	11.00	12.78	11.61	12.16	11.50	11.67	11.63	12.86
	(1.80)	(2.12)	(2.23)	(1.55)	(2.07)	(1.81)	(1.89)	(1.93)	(2.06)	(1.37)
6	11.37	12.61	12.27	13.00	11.51	11.26	11.14	12.36	10.41	11.90
	(2.19)	(1.69)	(1.58)	(1.46)	(2.02)	(2.25)	(2.05)	(1.63)	(2.39)	(1.96)
7									12.05	11.94
									(1.54)	(1.80)
					Rectangles					
K	8.91	10.05	9.59	10.80	10.12	10.11	9.85	10.85	8.89	10.88
	(2.98)	(1.67)	(2.18)	(2.00)	(1.63)	(1.59)	(1.44)	(1.74)	(1.88)	(1.45)
1	8.41	10.31	9.18	10.83	9.44	10.68	9.17	8.74	9.56	10.22
	(2.38)	(2.87)	(2.28)	(2.29)	(1.18)	(3.03)	(1.99)	(2.21)	(1.69)	(1.44)
2	9.60	10.78	8.58	9.72	8.69	10.15	8.78	8.51	9.96	10.95
	(2.41)	(2.55)	(2.15)	(2.51)	(2.31)	(2.19)	(2.34)	(2.55)	(1.68)	(1.86)
3	10.43	9.78	9.75	10.00	9.30	8.33	10.00	9.81	9.33	–
	(1.56)	(1.51)	(2.47)	(2.80)	(1.34)	(1.49)	(2.04)	(2.34)	(2.27)	–
4	9.78	11.09	9.14	11.28	9.75	11.11	9.24	8.97	10.41	11.08
	(2.33)	(2.98)	(2.64)	(2.64)	(2.08)	(2.56)	(2.12)	(1.95)	(1.73)	(2.06)
5	10.64	11.17	9.54	11.91	9.71	10.78	9.84	10.02	11.33	12.70
	(3.00)	(2.96)	(2.33)	(2.99)	(2.18)	(2.87)	(2.93)	(2.54)	(1.81)	(1.99)
6	10.19	10.48	9.84	12.11	10.49	10.51	9.39	10.84	11.00	10.85
	(2.35)	(2.82)	(2.59)	(3.00)	(2.58)	(2.58)	(2.66)	(2.88)	(1.37)	(1.97)
7									11.00	12.44
									(1.60)	(2.43)
					Squares					
K	11.57	14.20	13.03	13.88	12.65	13.00	13.53	13.59	9.74	14.00
	(4.28)	(1.28)	(2.81)	(1.07)	(1.70)	(1.63)	(1.08)	(.99)	(4.50)	(1.37)
1	13.66	14.05	13.62	13.74	13.81	13.92	13.72	13.70	12.87	13.37
	(1.48)	(1.10)	(1.03)	(1.58)	(1.18)	(1.15)	(1.31)	(1.15)	(2.78)	(2.15)
2	13.51	13.25	13.11	14.04	13.33	13.60	13.18	13.56	12.52	13.39
	(1.53)	(2.36)	(1.57)	(1.40)	(1.49)	(1.50)	(1.95)	(1.50)	(2.64)	(2.06)
3	13.39	12.09	13.42	13.35	12.67	12.57	13.65	13.95	14.08	–
	(1.27)	(2.19)	(1.56)	(1.72)	(2.31)	(1.94)	(1.23)	(1.12)	(.79)	–
4	13.20	13.58	13.30	13.14	13.58	14.14	13.10	13.44	13.22	13.80
	(1.35)	(1.89)	(1.75)	(2.51)	(1.80)	(1.40)	(1.71)	(1.39)	(2.11)	(1.63)
5	13.82	13.93	13.70	14.02	13.67	14.04	13.66	13.89	13.98	14.12
	(2.06)	(1.99)	(1.71)	(1.63)	(1.31)	(.89)	(1.31)	(1.42)	(1.25)	(1.61)
6	13.96	13.41	13.80	14.27	13.88	13.79	13.41	13.62	13.24	13.08
	(1.04)	(2.14)	(1.21)	(1.35)	(1.32)	(1.39)	(1.22)	(1.98)	(1.92)	(1.84)
7									13.63	14.56
									(1.34)	(.98)

Table 6
Means and Standard Deviations for Identifying Statements That Describe Each Type of Figure (Item PP/2.10)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
					Squares					
2	4.83	6.69	5.38	6.54	4.98	5.51	4.85	5.24	6.77	8.66
	(1.74)	(1.12)	(1.43)	(1.38)	(1.27)	(1.20)	(1.35)	(1.18)	(1.61)	(1.02)
3	4.30	6.30	3.75	6.78	3.50	2.95	3.18	5.05	6.33	
	(1.82)	(1.58)	(1.73)	(.90)	(1.57)	(.95)	(1.26)	(1.50)	(1.87)	
4	3.88	6.81	4.11	6.98	3.96	5.86	4.91	6.13	5.00	9.22
	(1.93)	(1.39)	(1.98)	(1.19)	(1.80)	(1.46)	(1.70)	(1.70)	(2.08)	(.82)
5	6.91	7.66	4.67	7.48	5.96	6.86	5.59	6.62	8.02	9.58
	(1.18)	(.57)	(1.90)	(1.05)	(1.46)	(1.06)	(1.80)	(1.67)	(1.62)	(1.01)
6	5.74	7.27	6.69	7.42	5.42	6.14	6.31	6.98	6.47	9.21
	(2.16)	(1.19)	(1.42)	(1.10)	(2.05)	(1.42)	(1.60)	(1.60)	(2.33)	(.93)
7									8.68	9.61
									(1.67)	(.78)
					Rectangles					
2	4.30	4.12	4.27	5.33	4.02	4.64	3.75	4.71	5.93	7.68
	(1.23)	(2.05)	(1.53)	(1.33)	(1.52)	(1.33)	(1.41)	(1.12)	(1.67)	(1.88)
3	3.57	4.96	3.75	5.57	3.70	3.27	3.45	4.30	6.17	
	(1.16)	(1.22)	(1.39)	(.99)	(1.15)	(1.03)	(1.06)	(1.34)	(1.75)	
4	3.57	5.90	3.61	5.43	3.68	5.14	4.07	5.13	5.33	8.96
	(1.28)	(1.37)	(1.38)	(1.34)	(1.31)	(1.36)	(1.47)	(1.49)	(1.72)	(1.65)
5	5.78	6.49	4.22	6.50	5.04	5.59	4.34	5.24	7.90	9.15
	(1.49)	(.78)	(1.47)	(.91)	(1.60)	(1.38)	(1.54)	(1.52)	(1.75)	(1.35)
6	4.64	5.89	5.43	6.16	4.46	5.14	4.60	5.86	6.27	8.51
	(1.55)	(1.28)	(1.54)	(1.21)	(1.69)	(1.61)	(1.70)	(1.48)	(1.96)	(1.68)
7									8.21	9.50
									(1.58)	(.99)

Table 7
Means and Standard Deviations for "What Shape Could I Be Thinking Of?" (Item PP/2.11)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
4	2.66	2.80	2.84	3.28	2.62	3.14	3.41	3.87	2.77	3.24
	(1.14)	(1.02)	(.82)	(1.12)	(.99)	(1.11)	(.84)	(.86)	(1.02)	(.72)
5	3.23	3.36	2.98	3.67	2.94	3.24	3.37	3.87	3.36	4.00
	(1.17)	(1.01)	(1.20)	(1.09)	(1.00)	(1.05)	(1.00)	(.99)	(.74)	(.93)
6	3.12	3.21	3.65	3.86	3.16	2.71	3.32	3.77	3.14	3.52
	(1.04)	(1.19)	(1.05)	(.99)	(1.32)	(1.17)	(1.04)	(.99)	(.80)	(.84)
7									3.79	4.13
									(.71)	(1.1)

Table 8
Means and Standard Deviations for Disembedding, K–1 (Item PP/2.6)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	5.17	6.70	5.43	6.54	2.85	6.42	5.56	6.53	–	–
	(1.87)	(.73)	(1.01)	(.74)	(2.11)	(1.02)	(1.08)	(.96)		
1	6.73	7.79	6.13	6.69	6.70	8.18	6.32	6.87	–	–
	(.90)	(1.42)	(1.01)	(.92)	(.89)	(1.45)	(.59)	(.34)		

Table 9
Means and Standard Deviations for K–3 Building (Item PP/2.A)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	.04	.05	.00	.05	.04	.11	.09	.00	.00	.00
	(.21)	(.23)	(.00)	(.23)	(.20)	(.32)	(.29)	(.00)	(.00)	(.00)
1	.18	.36	.02	.09	.07	.22	.09	.09	.00	.00
	(.39)	(.49)	(.15)	(.29)	(.26)	(.42)	(.28)	(.29)	(.00)	(.00)
2	.06	.24	.07	.22	.04	.11	.03	.46	.04	.13
	(.25)	(.43)	(.25)	(.42)	(.20)	(.31)	(.16)	(.50)	(.19)	(.34)
3	.00	.43	.00	.32	.03	.00	.04	.20	.08	
	(.00)	(.51)	(.00)	(.48)	(.18)	(.00)	(.21)	(.41)	(.29)	

Table 10
Triad Discrimination Scores

	LG (87–88)			*LG* (88–89)			Control (88–89)
Grade	Pre	Int	Post	Pre	Int	Post	Post (only)
K	.20	1.10	2.70	.00	.14	.00	.29
1	1.00	1.00	1.30	.00	1.38	.56	.25
2	.63	1.50	.88	.13	1.13	.88	.25
3	.67	.33	1.00	.50	2.63	1.50	.57
4	1.07	2.47	1.53	.21	1.84	1.74	1.70
5	1.82	3.55	3.64	1.61	5.50	4.39	3.94
6	1.60	4.00	4.10	3.95	5.76	6.76	5.30
7	4.88	6.50	7.25				

Table 11
Descriptions of Reasons for Choosing Each Triad Pair Classified by van Hiele Level

VISUAL	This is given if student's reason fits one of these.
look like object	both objects look like a window, door, or box
3D projection	student interprets drawings as pictures of 3D objects or such objects in perspective (e.g., This looks like the top of a box and this one the side of the box. This trapezoid looks like what you would get it you looked at a rectangle from here).
slanted sides	both have slanted sides (or, nonslanted sides)
sharp angles	both are "pointy" or have "sharp corners"
deformation	both would look the same if I moved this line over here (includes deformation)
size	both are the same size (big, little, fat, skinny, long, short)
same shape	both look like they are the same shape
look like shape	both look like (some specific geometric and abstract shape)
share characteristic	both share a visual characteristic or specific component
look same	both look the same or look alike
motions	both are the same, but one is moved, shrunk, stretched, flipped, turned, slid (or they would be the same if one was moved)
EXPPROP	**This is given if student's reason fits one of these**
same # sides	both have same number of sides
same # angles	both have the same number of angles (or turns or bends)
side length	both have *x* sides the same length or not the same length
parallel sides	both have a pair of opposite sides parallel
right angles	both have (or do not have) right angles (or 90° turns)
angle	other specific references to angle-related properties (e.g., having to do with relationships between measure of angles, such as "these both have acute angles")
symmetry	both are or are not symmetric
congruent/sim	both are congruent or similar
same #sides/ same #angles	mentioned having same number of sides and same number of angles with no intervening mention of another property
other property	names another property
EXPCLASS	**This is given if student's reason fits one of these**
classify	they both are parallelograms, rectangles, scalene, obtuse, isosceles, and right, etc. (Further justification by the student must not contradict this level.)
OTHER	**This is given if student's reason fits no other category**
other	any other reason

Table 12
Percent of Students Choosing Categories of Reasons for Triads

Triad	1	2	3	4	5	6	7	8	9	Mean
					Properties or classification reasons					
pretest	16	9	6	18	3	11	9	13	9	10
posttest	42	28	16	42	12	21	20	27	14	25
increase	26	*18	10	24	10	10	11	15	5	15
					Visual reasons					
pretest	84	86	90	80	95	85	85	82	86	86
posttest	55	65	78	44	80	71	77	63	76	68
decrease	29	22	12	36	15	14	9	19	10	18

* pre and post may not give exact increase or decrease because all figures are rounded
Note. Because the triad items were different for the first year, these items are analyzed only for the 1988–89 data.

Table 13

Percent of Students at Each van Hiele Level for Pretest (1), Interim (2), and Posttest (3) Interviews

	Grade																							
	K			1			2			3			4			5			6			7		
Level*	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
Logo Site 1 1987–88																								
0	10	30	20	10																				
0.5		20		10	10	10	13				17		7					9	10					
1	90	50	60	80	90	80	88	88	88	100	83	100	87	73	87	91	73	55	80	80	60	43	14	25
1.5			20			10		13	13				7	27	13	9	27	27		10	30	43	86	38
2																		10			14			
2.5																		9		10	10			13
3																								25
Logo Site 1 1988–89																								
0			100																					
0.5	33	100		50	67							100	29	22		20		29						
1	67			50	33	100	100	75	50	100	100		71	56	75	20	40	29	40	33	27			
1.5								25	50					22	25	40	20	14	50	25	27			
2																	40	29	10	42				
2.5																20					46			
3																								
Logo Site 2 1988–89																								
0			50			25	17		50					20	33									
0.5				29	33		17	67	50			67		20	33		20				20			
1	100	100	50	71	67	50	67	33		100	80	33	100	20	33	100	20	33	71	67	20			
1.5						25					20			40			60	67	29		40			
2																								
2.5																				33	20			
3																								

* Fractional values indicate students in transition to the next higher level.

Note. For each grade at each site, this table presents the percent of students classified at each level of the van Hiele hierarchy. For example, at Logo Site 1 1987–88, kindergarten, at the pretest (interview 1 on this table), 10% were classified at Level 0, 90% at Level 1. By the posttest interview (3), 20% were at Level 0, 60% at Level 1, and 20% in transition to Level 2 (i.e., Level 1.5).

Table 14
Percent of Students by Grade Level Whose van Hiele Level Increased from Pre- to Post-Interview

Grade	*LG*(87–88)	*LG*(88–89 Site 1)	*LG*(88–89 Site 2)
K	30.0	0	0
1	30.0	0	0
2	25.0	0	0
3	0	25.0	16.7
4	12.5	8.3	14.3
5	36.4	44.4	25.0
6	50.0	46.2	37.5
7	85.7		

Table 15
Percent of Responses for Reasoning About Quadrilaterals

Team	Pre	Interim	Post	Control
		Team choices		
Site 1				
A	52.8	14.3	5.4	21.9
B	27.8	51.4	54.1	43.8
C	8.3	20.0	16.2	15.6
D	2.8	5.7	2.7	3.1
E	2.8	2.9	0.0	6.3
A&C		2.9	5.4	3.1
B, D&E	2.8		5.4	
B&D	2.8		5.4	
D&E		2.9	5.4	
B&E				6.3
Site 2				
A	33.3	12.0	12.5	12.5
B	29.2	60.0	58.3	54.2
C	8.3	12.0	8.3	12.5
D	20.8	8.0	12.5	8.3
E	8.3	8.0	8.3	12.5
		Student choices for teams that would make a rectangle		
Site 1				
A	63.9	22.9	13.5	28.1
B	91.7	88.6	97.3	90.6
C	13.9	28.6	27.0	18.8
D	25.0	34.3	45.9	25.0
E	22.2	25.7	37.8	31.3
Site 2				
A	54.2	50.0	54.2	45.8
B	83.3	96.0	95.8	83.3
C	12.5	36.0	25.0	33.3
D	37.5	54.2	54.2	75.0
E	41.7	58.3	62.5	58.3

Table 16
Percent by Category in Reponse to "What Is an Angle?" (Item PP/2.1)

Category	*LG* (1988–89)		Control (1988–89)	
	Pre	Post	Pre	Post
No interpretable response	38	14	38	22
Shape or part of a shape	6	6	7	9
Line/segment/side (e.g., of a square)	2	3	4	2
Tilted line	11	4	10	8
Orientation/heading	2	0	2	1
Rotation/turn	2	19	2	1
Something that forms a square corner	5	5	4	3
Intersection of two lines/segments	8	15	8	20
Union of two lines/segments	5	4	7	9
Looks like object (e.g., a half of a boat)	1	3	1	2
Something that bends	4	7	4	3
A thing with degrees (measured)	3	4	4	4
Special path (as in curriculum)	0	1	0	0
A line that is not straight	2	1	1	1
Something that is closed or not closed	0	1	0	0
Other	11	14	9	15

Table 17
Means and Standard Deviations for "Circle All the Angles" (Item PP/2.5)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	7.52	11.05	7.54	6.71	7.15	7.47	8.68	8.15	6.95	11.71
	(1.95)	(1.54)	(2.52)	(3.11)	(1.85)	(2.99)	(2.36)	(2.86)	(2.30)	(1.76)
1	7.20	10.95	8.13	9.98	7.49	10.76	8.87	10.74	7.04	9.56
	(2.02)	(2.26)	(2.55)	(2.11)	(1.92)	(2.63)	(2.17)	(1.83)	(2.49)	(2.69)
2	8.53	10.65	8.16	9.91	8.63	11.00	8.33	8.59	7.18	11.03
	(2.22)	(2.57)	(2.29)	(2.21)	(2.12)	(1.99)	(2.14)	(2.38)	(2.23)	(2.39)
3	8.96	9.43	7.17	10.52	7.07	8.68	7.32	11.20	7.08	
	(1.58)	(2.39)	(3.03)	(1.90)	(2.86)	(2.77)	(2.12)	(1.15)	(1.88)	
4	9.19	11.14	8.98	11.78	9.17	11.98	9.82	11.28	8.67	11.63
	(2.77)	(2.05)	(2.47)	(1.33)	(2.63)	(1.38)	(2.30)	(1.47)	(2.67)	(1.72)
5	11.31	12.07	10.57	11.11	11.33	12.45	10.80	11.64	11.31	12.35
	(2.07)	(1.44)	(2.12)	(2.09)	(2.31)	(1.32)	(2.21)	(1.72)	(2.00)	(1.05)
6	10.97	11.52	11.96	12.16	11.30	11.86	11.45	12.16	10.49	11.55
	(2.78)	(2.04)	(1.29)	(1.83)	(2.61)	(2.80)	(2.32)	(2.03)	(2.27)	(2.23)
7									12.21	12.72
									(1.36)	(1.53)

Table 18
Means and Standard Deviations for "Draw an Angle" (Item PP/2.2)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	.00	.75	.14	.23	.04	.11	.24	.32	.00	.82
	(.00)	(.44)	(.35)	(.43)	(.20)	(.32)	(.43)	(.47)	(.00)	(.39)
1	.14	.74	.33	.60	.14	.82	.34	.78	.07	.54
	(.35)	(.44)	(.48)	(.50)	(.35)	(.39)	(.48)	(.42)	(.25)	(.50)
2	.13	.76	.27	.72	.06	.72	.00.	.24	.11	.53
	(.34)	(.43)	(.45)	(.46)	(.24)	(.45)	(.00)	(.43)	(.31)	(.51)
3	.22	.74	.21	.87	.20	.45	.27	.85	.00	
	(.42)	(.45)	(.41)	(.34)	(.41)	(.51)	(.46)	(.37)	(.00)	
4	.51	.88	.50	.98	.46	.88	.66	1.00	.36	.88
	(.50)	(.33)	(.51)	(.16)	(.50)	(.33)	(.48)	(.00)	(.48)	(.33)
5	.89	1.00	.85	.89	.86	.98	.91	.87	.92	1.00
	(.32)	(.00)	(.36)	(.31)	(.35)	(.14)	(.29)	(.34)	(.28)	(.00)
6	.70	.89	.94	1.00	.85	.86	1.00	.96	.69	.94
	(.46)	(.32)	(.24)	(.00)	(.36)	(.35)	(.00)	(.20)	(.47)	(.25)
7									1.00	1.00
									(.00)	(.00)

Table 19
Means and Standard Deviations for "Draw a Bigger Angle" (Item PP/2.3)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	.00	.20	.00	.06	.04	.00	.06	.06	.00	.00
	(.00)	(.41)	(.00)	(.24)	(.20)	(.00)	(.24)	(.24)	(.00)	(.00)
1	.00	.08	.04	.12	.00	.11	.06	.26	.00	.00
	(.00)	(.27)	(.21)	(.33)	(.00)	(.31)	(.25)	(.44)	(.00)	(.00)
2	.02	.18	.04	.26	.04	.38	.00	.07	.02	.03
	(.15)	(.39)	(.21)	(.44)	(.20)	(.49)	(.00)	(.26)	(.13)	(.16)
3	.04	.22	.04	.39	.00	.23	.00	.10	.00	
	(.21)	(.42)	(.20)	(.50)	(.00)	(.43)	(.00)	(.31)	(.00)	
4	.20	.59	.11	.70	.16	.29	.23	.46	.09	.39
	(.40)	(.50)	(.32)	(.46)	(.37)	(.46)	(.42)	(.51)	(.29)	(.49)
5	.60	.83	.28	.74	.43	.76	.50	.80	.33	.03
	(.50)	(.38)	(.46)	(.44)	(.50)	(.43)	(.51)	(.40)	(.48)	(.16)
6	.47	.68	.73	.91	.42	.60	.65	.88	.27	.72
	(.50)	(.47)	(.45)	(.29)	(.50)	(.49)	(.48)	(.33)	(.45)	(.45)
7									.42	.94
									(.00)	(.24)

Table 20
Means and Standard Deviations for the Spinner or Amount of Turn (Item PP/2.4)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
2	.00	1.45	.09	1.33	.12	.64	.18	.17	.00	1.39
	(.00)	(.83)	(.29)	(.82)	(.44)	(.85)	(.45)	(.44)	(.00)	(.86)
3	.00	1.22	.00	2.09	.00	.18	.00	1.35		
	(.00)	(.80)	(.00)	(.85)	(.00)	(.50)	(.00)	(.59)		
4	.18	1.76	.25	1.65	.26	.55	.16	.92	.21	1.82
	(.44)	(1.09)	(.65)	(.92)	(.60)	(.84)	(.57)	(.98)	(.57)	(.86)
5	1.07	2.14	.39	2.26	.65	1.04	.45	.89	.73	2.23
	(1.18)	(.81)	(.80)	(.77)	(1.00)	(1.00)	(.66)	(.78)	(.98)	(.97)
6	.83	2.00	.90	2.42	.80	1.37	.55	1.18	.33	2.02
	(1.06)	(.96)	(.96)	(.78)	(1.04)	(1.5)	(.79)	(1.01)	(.72)	(1.01)
7									.84	2.22
									(.83)	(1.17)

Table 21
Means and Standard Deviations for "Estimate the Measure of Each Angle" (Item PP/1.7)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
3	1.41	2.05	1.82	2.43			1.50	1.75	1.25	
	(.96)	(1.24)	(1.30)	(1.16)			(.96)	(.91)	(.62)	
4	1.46	2.83	1.71	3.08	1.62	2.67	1.71	2.94	1.77	2.73
	(1.01)	(1.14)	(1.21)	(1.00)	(.97)	(1.03)	(1.07)	(1.17)	1.18)	(.98)
5	2.65	3.51	2.08	3.11	2.32	3.08	2.40	3.56	2.37	3.19
	(1.27)	(.81)	(1.11)	(.97)	(1.02)	(1.17)	(1.38)	(.76)	1.33)	(1.01)
6	2.95	2.95	3.22	3.50	2.66	3.13	3.27	3.70	2.59	3.27
	(1.24)	(.99)	(1.03)	(.85)	(1.22)	(1.22)	(.99)	(.61)	1.27)	(.82)
7									3.79	3.89
									(.63)	(.32)

Table 22
Means and Standard Deviations for "Find the Missing Measures of the Angles and of the Sides" (Items PP/1.8–1.9)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
4	1.55	2.68	1.32	2.56	1.56	1.80	2.43	2.44	1.60	2.73
	(1.38)	(1.43)	(1.28)	(1.66)	(1.37)	(1.34)	(1.47)	(.97)	(1.30)	(1.32)
5	2.63	4.83	2.16	4.52	2.67	3.90	2.10	3.53	2.31	4.30
	(1.33)	(1.95)	(1.52)	(1.67)	(1.14)	(1.67)	(1.62)	(2.05)	(1.46)	(1.70)
6	2.49	3.86	2.96	4.93	2.41	3.03	2.55	4.62	2.24	4.52
	(1.69)	(1.81)	(1.40)	(1.93)	(1.67)	(2.22)	(1.25)	(1.75)	(1.46)	(1.89)
7									3.32	6.06
									(1.46)	(1.66)

Table 23
Means and Standard Deviations for "Angle Measure: Boat Amount-of-Turn" (Item PP/2.12)

	LG (1988–89)				Control (1988–89)			
	Site 1		Site 2		Site 1		Site 2	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post
4	.05	.10	.05	.05	.00	.04	.05	.03
	(.22)	(.30)	(.22)	(.23)	(.00)	(.19)	(.22)	(.17)
5	.12	.29	.00	.23	.05	.05	.15	.06
	(.33)	(.46)	(.00)	(.43)	(.22)	(.21)	(.37)	(.24)
6	.12	.29	.09	.48	.13	.10	.07	.26
	(.33)	(.46)	(.29)	(.51)	(.33)	(.30)	(.25)	(.44)

Table 24
Means and Standard Deviations for "Angle Measure: Bending Wire" (Item PP/2.13)

	LG (1988–89)				Control (1988–89)			
	Site 1		Site 2		Site 1		Site 2	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post
2	.00	.49	.00	.26	.00	.02	.03	.06
	(.00)	(.51)	(.00)	(.44)	(.00)	(.15)	(.16)	(.24)
3	.00	.14	.00	.48	.00	.00	.00	.21
	(.00)	(.35)	(.00)	(.51)	(.00)	(.00)	(.00)	(.43)
4	.15	.28	.08	.42	.04	.14	.03	.08
	(.36)	(.45)	(.28)	(.50)	(.20)	(.35)	(.17)	(.28)
5	.26	.56	.13	.53	.13	.16	.12	.32
	(.44)	(.50)	(.34)	(.51)	(.34)	(.37)	(.33)	(.47)
6	.10	.44	.22	.86	.19	.15	.20	.66
	(.30)	(.50)	(.42)	(.35)	(.40)	(.36)	(.40)	(.48)

Table 25
Classifications of Students' Angle Drawings (Percent of Students)

	LG									Control (88–89)	
	Site 1 (88–89)			Site 2 (88–89)			Site 1 88			Site 1	Site 2
	Pre	Int	Post	Pre	Int	Post	Pre	Int	Post	Post	Post
None	20.3	4.9	0	11.1	3.6	0	13	12.5	2	11.1	9.4
Segment	3.1	0	1.6	0	0	1.9	1	0	0	3.7	1.9
Shape	9.4	11.5	7.9	11.1	9.1	5.6	9	12.5	6	11.1	3.8
Tilted seg.	15.6	8.2	11.1	20.4	7.3	9.3	25	5	9	7.4	13.2
Intersection	0	0	0	0	0	0	1	0	0	0.0	0.0
Heading	0	0	0	0	0	0	0	0	0	0.0	0.0
Object	0	0	1.6	0	0	0	2	0	1	0.0	1.9
Arrow	0	0	0	0	0	0	0	0	0	0.0	0.0
Arc	0	0	1.6	3.7	0	0	1	1.2	0	1.9	1.9
Total Incor.	**48.4**	**24.6**	**23.8**	**46.3**	**20.0**	**16.8**	**52**	**31.2**	**18**	**35.2**	**32.1**
Proto. right	9.4	11.5	7.9	18.5	20.0	22.2	9	18.8	17	22.2	22.6
Proto. acute	7.8	6.6	3.2	9.3	5.5	1.9	7	6.3	4	18.5	18.9
Proto. obtuse	0	0	3.2	1.9	1.8	3.7	1	0	3	3.7	3.8
subtotal	17.2	18.1	14.3	29.7	27.3	27.8	17	25.1	24	44.4	45.3
Right	12.5	23.0	15.9	0	10.9	9.3	9	8.7	13	0.0	5.7
Acute	17.2	27.9	36.5	16.7	32.7	44.4	20	32.5	40	16.7	11.3
Obtuse	3.1	4.9	7.9	0	5.5	1.9	2	2.5	5	3.7	0.0
Straight	0	0	1.6	0	1.8	0	0	0	0	0.0	1.9
subtotal	32.8	55.8	61.9	16.7	50.9	55.6	31	43.7	58	20.4	17.9
Total correct	**50.0**	**73.9**	**76.2**	**46.4**	**78.2**	**83.4**	**48**	**68.8**	**82**	**64.8**	**64.2**
Other	1.6	1.6	0	7.4	1.8	0	0	0	0	0.0	5.7

Table 26
Percents of Classifications of Students' Responses to "How Is This Angle Different From This One?"

	LG									Control (88–89)	
	Site 1 (88–89)			Site 2 (88–89)			Site 1 (87–88)			Site 1	Site 2
	Pre	Int	Post	Pre	Int	Post	Pre	Int	Post	Post	Post
None	21.9	4.9	0.0	2.0	0.0	0.0	1.2	0.0	0.0	10.9	0.0
Length of sides	1.6	8.2	14.5	12.2	5.5	5.6	14.8	17.4	25.9	3.6	5.7
Shape	32.9	21.3	21.0	24.5	5.4	11.1	30.8	22.5	22.2	14.5	13.2
Number of sides	6.2	8.2	1.6	10.2	3.6	1.9	6.2	8.7	4.9	9.1	0.0
No understanding	62.6	42.6	37.1	48.9	14.5	18.6	53.0	48.6	53.0	38.1	18.9
Tilt	17.2	16.4	8.1	16.3	10.9	13.0	24.7	8.7	8.6	9.1	11.3
Heading	34.4	29.5	30.6	22.4	34.5	33.3	22.3	26.2	37.0	29.1	22.6
Turn on path	0.0	4.9	4.8	2.0	1.8	1.9	0.0	3.7	2.5	0.0	1.9
Related	51.6	50.8	43.5	40.7	47.2	48.2	47.0	38.6	48.1	38.2	35.8
Size (unelab.)	25.0	21.3	41.9	20.4	32.8	27.8	12.3	17.5	39.5	23.6	26.4
Wider	9.4	23.0	12.9	12.2	14.5	29.6	19.8	22.5	24.7	20.0	13.2
Sharper pt.	1.6	0.0	1.6	10.2	0.0	3.8	—	—	—	0.0	1.9
Intuitive	36.0	44.3	56.4	42.8	47.3	61.2	32.1	40.0	64.2	43.6	41.5
Rotation/rays	1.6	3.3	3.2	0.0	1.8	0.0	0.0	2.5	0.0	1.8	0.0
Measure	14.1	32.8	19.4	10.2	16.3	25.9	9.9	35.0	22.2	18.2	24.5
Classify	9.4	21.3	32.3	8.1	23.7	29.6	6.2	18.7	29.6	36.3	32.1
Formal	25.1	57.4	54.9	18.3	41.8	55.5	16.1	56.2	51.8	56.3	56.6
Deform	0.0	0.0	0.0	0.0	0.0	0.0	2.4	0.0	1.2	0.0	0.0
Other	12.5	9.8	12.9	6.1	14.9	22.6	6.1	11.2	9.9	7.3	24.5

Note. Underscored terms are summaries.

Table 27
Angle Scores by Grade (Out of a Possible 4)

	LG (87–88)			*LG* (88–89)			Control (88–89)
Grade	Pre	Int	Post	Pre	Int	Post	Post
K	.30	.40	.60	.07	.21	.21	.00
1	.00	.50	.50	.31	.44	.56	.50
2	.25	1.25	1.50	.38	1.19	1.31	.13
3	.00	.33	.17	.70	1.90	2.20	.64
4	.87	2.73	2.33	.74	2.84	2.63	2.25
5	2.18	2.82	2.82	1.78	3.44	3.33	3.06
6	2.50	3.00	3.50	2.86	3.48	3.67	3.15
7	3.38	3.75	3.88				

Table 28
Choices for the Question "Which Angle Is Bigger?" (Percent of Students)

	Site 1 89			Site 2 89			Site 1 88			Control 89 post	
	Pre	Int	Post	Pre	Int	Post	Pre	Int	Post	Site 1	Site 2
Pair 1											
IDK*	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0	0.0	3.6	0.0
A	40.7	62.5	61.7	39.1	67.2	66.7	22.2	58.2	50.0	47.3	49.1
B	59.3	37.5	38.3	59.4	32.8	31.7	77.8	40.0	48.1	47.3	50.9
Same	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.8	1.9	1.8	0.0
Pair 2											
IDK	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0	0.0	1.8	0.0
A	29.6	56.3	55.6	26.6	54.1	55.6	14.8	49.1	48.1	40.0	41.5
B	70.4	43.8	44.4	71.9	45.9	44.4	83.3	49.1	48.1	58.2	58.5
Same	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.8	3.7	0.0	0.0
Pair 3											
IDK	1.2	0.0	0.0	0.0	0.0	0.0	1.9	1.8	0.0	0.0	3.8
A	51.9	41.2	39.5	48.4	24.6	27.0	53.7	38.2	40.7	45.5	35.8
B	30.9	40.0	33.3	25.0	21.3	25.4	14.8	18.2	11.1	14.5	30.2
Same	16.0	18.8	27.2	26.6	54.1	47.6	29.6	40.0	46.3	40.0	28.3
Triangle											
IDK	3.7	8.7	1.2	1.6	1.6	4.8	0.0	1.8	1.9	7.3	3.8
A	7.4	3.7	4.9	18.8	19.7	9.5	9.3	18.2	18.5	14.5	17.0
B	1.2	1.2	2.5	3.1	4.9	1.6	0.0	9.1	5.6	3.6	11.3
C	35.8	61.2	58.0	34.4	54.1	58.7	18.5	38.2	57.4	41.8	35.8
side AB	27.2	17.5	9.9	29.7	13.1	19.0	42.6	18.2	9.3	23.6	26.4
side BC	2.5	0.0	0.0	0.0	0.0	0.0	0.0	1.8	0.0	0.0	0.0
side AC	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0	0.0	1.8	0.0
whole figure	22.2	7.5	23.5	10.9	4.9	1.6	24.1	10.9	7.4	7.3	3.8

* For each pair of angles, the figure on the left is "A" and the one on the right is "B." IDK = "I don't know."

Table 29
Choices for the Question "Which Angle Is Bigger?" by Grade (88–89 LG), Angle Pair 2

	Pretest		Interim		Posttest	
Choice	A	B	A	B	A	B
K	0	92.9	0	100	7.1	92.9
1	0	100	0	100	12.5	81.3
2	0	100	25	75	25	75
3	0	100	50	50	40	60
4	10	90	70	30	68.4	31.6
5	38.9	55.6	89.5	5.3	81	14.3
6	72.7	27.3	95.2	4.8	95.2	4.8

Table 30
Classifications of Students' Responses to "Why Did You Say This Angle Was Bigger?" (Percent of Students)

	Site 1 89			Site 2 89			Site 1 88			Control 89 post	
	Pre	Int	Post	Pre	Int	Post	Pre	Int	Post	Site 1	Site 2
None	1.6	3.3	0.0	0.0	0.0	0.0	0.0	2.5	1.2	0.0	0.0
Length of sides	56.3	39.3	34.9	66.7	38.2	27.8	69.1	45.0	50.6	54.6	39.6
Shape	7.8	1.6	3.2	9.3	0.0	0.0	6.2	7.5	2.5	5.5	3.8
Number of sides	3.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0
No understanding	68.8	44.2	38.1	76.0	38.2	27.8	75.3	55.0	55.5	60.1	43.4
Tilt	14.1	0.0	1.6	0.0	0.0	1.9	7.4	0.0	0.0	1.8	0.0
Heading	1.6	0.0	3.2	1.9	1.8	1.9	7.4	1.2	2.5	1.8	1.9
Turn on path	3.2	14.8	7.9	0.0	1.8	3.7	1.2	3.7	3.7	1.8	1.9
Related	18.9	14.8	12.7	1.9	3.6	7.5	16.0	4.9	6.2	5.4	3.8
Size (unelab.)	31.2	19.7	23.8	22.3	7.2	13.0	13.6	12.4	22.2	21.8	11.3
Wider	57.8	57.4	52.4	24.1	52.7	50.0	49.4	57.5	59.2	49.1	50.9
Sharper pt.	3.1	1.6	1.6	3.8	0.0	3.8	—	—	—	3.6	3.8
Intuitive	92.1	78.7	77.8	50.2	59.9	66.8	63.0	69.9	81.4	74.5	66.0
Rotation/rays	0.0	4.9	7.9	1.9	0.0	0.0	0.0	3.7	2.5	1.8	0.0
Measure	17.2	24.6	22.2	5.6	10.9	14.8	8.6	16.2	20.9	20.0	9.4
Classify	4.7	6.6	23.8	1.9	5.5	5.6	3.7	7.5	17.3	14.5	7.5
Formal	21.9	36.1	53.9	9.4	16.4	20.4	12.3	27.4	40.7	36.3	16.9
Deform	0.0	0.0	0.0	3.8	0.0	0.0	2.4	3.7	0.0	0.0	0.0
Other	9.4	11.5	4.8	4.0	7.3	20.7	3.7	5.0	2.4	5.5	11.6

Note. Underscored terms are summaries.

Table 31
Means and Standard Deviations for "Draw a Path" (Items PP/2.7–2.9)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
2	.71	2.16	.26	2.09	.41	1.74	.44	.29	.52	2.05
	(.61)	(.89)	(.51)	(.83)	(.67)	(1.04)	(.73)	(.58)	(.70)	(.72)
3	.35	2.09	.25	2.22	1.13	1.02	.45	.88	.75	
	(.63)	(1.08)	(.61)	(.80)	(.84)	(.98)	(.62)	(.86)	(.84)	
4	.74	2.12	.75	2.11	.50	1.01	.61	1.23	.90	2.26
	(.92)	(.76)	(1.01)	(.84)	(.73)	(1.02)	(.95)	(1.02)	(.89)	(.82)
5	2.01	2.44	.77	2.59	1.46	1.86	.61	1.12	1.76	2.44
	(.88)	(.48)	(1.07)	(.62)	(.89)	(.81)	(.84)	(.98)	(.93)	(.52)
6	1.30	2.61	1.19	2.37	.91	1.38	.85	1.32	1.43	2.11
	(1.05)	(.50)	(.92)	(.80)	(.89)	(1.11)	(1.01)	(.89)	(.85)	(.90)
7									2.08	1.92
									(.56)	(.43)

Table 32
Percents by Category for "Telephone" (Item PP/1.10)

Category	*LG* (1988–89) Pre	*LG* (1988–89) Post	Control (1988–89) Pre	Control (1988–89) Post
	Response			
No response	28	6	28	9
Different figure	2	3	3	4
Components only	54	50	51	56
Correct, but unwarranted assumptions or inexact measures	12	29	16	23
Correct, complete, procedure	4	13	3	8
	Language			
Indistinguishable	19	7	19	10
Specific example	7	3	6	5
Other	3	1	4	4
Simple path (turn, go up)	7	5	5	3
Components (lines, points)	60	62	59	75
Logo-like commands	0	10	0	0
Complex path or components + Logo	5	12	7	3

Table 33
Means and Standard Deviations for "Drawing Symmetry Lines" (Item PP/1.5)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
	Total first four items									
K	.30	3.89	.46	2.91	.08	.05	.47	.82	1.16	3.94
	(.70)	(1.82)	(1.02)	(1.50)	(.39)	(.23)	(.90)	(1.00)	(1.34)	(1.52)
1	.07	5.62	.93	4.81	.09	3.05	.34	3.07	.09	5.51
	(.46)	(1.21)	(1.16)	(1.85)	(.61)	(1.06)	(1.18)	(2.36)	(.42)	(1.50)
2	1.36	5.59	2.16	5.84	.47	4.33	1.43	3.63	1.29	4.92
	(1.75)	(2.48)	(2.20)	(1.96)	(1.42)	(1.65)	(1.77)	(.86)	(1.70)	(2.45)
3	2.57	5.26	2.42	5.39	1.87	2.48	2.78	5.10	2.17	
	(1.20)	(1.25)	(1.79)	(1.31)	(1.57)	(1.60)	(1.31)	(1.89)	(2.17)	
4	2.80	6.63	3.66	5.58	2.92	6.44	3.71	3.76	2.68	6.65
	(2.22)	(1.48)	(1.78)	(1.92)	(2.09)	(1.56)	(1.15)	(1.50)	(2.06)	(1.25)
5	5.11	6.90	2.07	7.15	5.84	7.37	3.18	5.69	5.90	7.26
	(2.78)	(1.54)	(2.47)	(1.09)	(1.65)	(1.17)	(1.78)	(1.41)	(1.77)	(1.00)
6	4.09	6.09	4.16	6.43	3.84	4.12	4.29	5.50	2.88	6.15
	(2.80)	(1.36)	(1.69)	(1.74)	(3.09)	(2.72)	(1.44)	(1.74)	(2.58)	(1.95)
7									6.16	8.00
									(1.89)	(.00)
	Total of all questions									
3	1.74	3.00	1.75	3.96	1.17	1.71	1.87	4.00	1.58	
	(1.05)	(1.45)	(1.45)	(1.55)	(1.21)	(1.35)	(1.29)	(1.84)	(1.73)	
4	2.20	5.71	2.45	4.47	2.24	5.91	2.69	2.82	2.01	5.82
	(2.00)	(2.01)	(1.56)	(2.13)	(1.80)	(1.96)	(.87)	(1.25)	(1.81)	(1.97)
5	4.32	6.17	1.41	6.59	4.86	6.63	2.16	3.87	5.25	6.98
	(2.72)	(2.06)	(1.69)	(1.50)	(1.79)	(1.65)	(1.27)	(2.05)	(2.11)	(1.34)
6	3.50	5.25	2.94	4.86	3.13	3.35	2.98	4.06	2.17	5.25
	(2.74)	(1.92)	(1.43)	(2.58)	(2.81)	(2.54)	(1.51)	(1.96)	(2.08)	(2.51)
7									5.32	7.67
									(2.00)	(.69)

Table 34
Percentage of Correct and Incorrect Symmetry Lines Drawn (Item PP/1.5.)

Lines Drawn	*LG* (1988–89)				Control (1988–89)			
	Correct Lines		Incorrect Lines		Correct Lines		Incorrect Lines	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Heart								
0	40	3	85	92	40	13	85	93
1	60	97	13	7	60	87	11	6
2			1				1	1
3			1	1			1	
4			1				1	
5–9			1				1	
Rectangle								
0	41	2	78	62	44	14	77	75
1	40	19	10	2	40	42	11	6
2	18	79	8	35	16	44	6	18
4			2	1			4	1
5–9			2				2	
incorrectly								
Pentagon								
0	48	10	76	91	48	23	76	84
1	52	90	18	7	52	77	16	14
2			2	0			3	1
3			1	1			1	
4			0				1	
5–9			3	1			4	1
Square								
0	37	3	90	98	38	11	89	95
1	43	15	6		44	43	6	3
2	8	17	1	1	9	12	1	1
3	1	2	0		1	1	1	1
4	12	63	2	4	9	32	2	
5–9			1				1	
Parallelogram								
0			31	39			19	34
1			55	29			65	43
2			9	22			11	15
3			1	2			2	2
4			4	8			3	6

Table 35
Means and Standard Deviations for "Draw the Other Half" (Item PP/1.6)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
K	.00	.25	.00	.34	.00	.00	.03	.13	.00	.97
	(.00)	(.41)	(.00)	(.48)	(.00)	(.00)	(.12)	(.35)	(.00)	(.37)
1	.08	.88	.31	.89	.21	.64	.19	.89	.12	.88
	(.22)	(.57)	(.55)	(.69)	(.43)	(.69)	(.46)	(.73)	(.29)	(.52)
2	.52	.95	.40	.95	.58	.75	.45	.95	.67	1.16
	(.57)	(.65)	(.59)	(.52)	(.58)	(.60)	(.53)	(.60)	(.58)	(.49)
3	.52	.72	.98	1.43	.47	.67	1.57	1.45	.96	
	(.51)	(.52)	(.91)	(.55)	(.47)	(.68)	(.51)	(.69)	(.50)	
4	.75	1.50	1.11	1.53	.97	1.37	1.33	1.42	1.03	1.50
	(.65)	(.53)	(.71)	(.51)	(.65)	(.38)	(.71)	(.65)	(.62)	(.46)
5	1.33	1.61	1.51	1.78	1.53	1.45	1.56	1.66	1.24	1.65
	(.65)	(.52)	(.67)	(.36)	(.48)	(.48)	(.62)	(.44)	(.47)	(.42)
6	1.24	1.60	1.81	1.93	1.04	1.29	1.63	1.77	1.05	1.36
	(.58)	(.45)	(.48)	(.23)	(.62)	(.38)	(.53)	(.41)	(.56)	(.47)
7									1.47	1.67
									(.42)	(.38)

Table 36
Means and Standard Deviations for "Congruency—Are Figures the Same Size and Shape?" (Item PP/1.4)

	LG (1988–89)				Control (1988–89)				*LG* (1987–88)	
	Site 1		Site 2		Site 1		Site 2		Site 1	
Grade	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
					Identification					
K	8.43	9.80	8.24	9.43	7.73	8.94	8.85	9.00	4.53	5.76
	(1.24)	(.89)	(1.34)	(1.09)	(1.43)	(1.00)	(1.33)	(1.17)	(1.17)	(.56)
1	8.93	9.67	9.00	9.93	8.67	9.34	8.72	9.32	4.71	5.71
	(1.21)	(1.08)	(1.29)	(1.03)	(1.21)	(1.05)	(1.14)	(.80)	(.94)	(.56)
2	9.38	9.66	9.34	9.48	9.60	9.57	9.19	9.10	4.95	5.68
	(1.29)	(1.53)	(1.24)	(1.05)	(1.25)	(1.30)	(1.03)	(1.36)	(.70)	(.57)
3	9.30	8.83	9.13	9.48	8.83	8.76	9.70	9.76	4.50	
	(1.29)	(1.85)	(.95)	(.85)	(1.39)	(2.17)	(1.15)	(.70)	(1.45)	
4	9.26	9.53	8.84	8.97	8.99	9.33	9.38	9.05	5.13	5.78
	(1.46)	(.94)	(1.31)	(1.84)	(1.16)	(.91)	(1.01)	(1.34)	(1.24)	(.51)
5	9.09	9.76	9.35	9.67	9.53	9.55	9.07	9.49	5.54	5.65
	(1.41)	(.69)	(.74)	(1.06)	(.90)	(.65)	(1.09)	(.94)	(.97)	(.75)
6	9.47	9.57	9.71	9.70	9.21	9.33	9.29	9.68	5.22	5.33
	(1.27)	(1.15)	(.91)	(1.00)	(1.73)	(1.02)	(.97)	(.87)	(1.09)	(1.06)
7									5.89	5.83
									(.32)	(.38)
					Justification					
K	1.09	5.00	3.05	4.05	3.27	3.17	3.68	4.21	2.47	3.71
	(1.38)	(1.34)	(2.22)	(1.87)	(2.13)	(1.72)	(2.38)	(1.27)	(1.61)	(1.36)
1	2.41	5.08	4.07	4.96	3.44	4.50	4.22	4.61	3.40	4.22
	(2.05)	(1.22)	(1.33)	(1.36)	(1.98)	(2.00)	(1.44)	(1.13)	(1.68)	(1.49)
2	4.02	5.41	3.88	5.14	4.33	4.64	4.17	4.29	3.21	4.24
	(2.03)	(1.06)	(1.66)	(1.11)	(1.78)	(1.42)	(1.75)	(1.64)	(1.79)	(1.44)
3	1.96	5.26	.29	5.13	2.13	2.81	.32	2.19		2.50
	(2.36)	(1.48)	(.55)	(1.36)	(1.93)	(2.27)	(1.29)	(2.29)		(2.43)
4	3.83	5.46	4.40	4.44	3.52	4.48	4.41	3.56	2.67	4.12
	(2.25)	(1.20)	(1.47)	(1.98)	(2.34)	(2.15)	(1.68)	(2.02)	(1.81)	(1.27)
5	4.75	5.55	4.67	5.46	4.94	5.29	4.39	5.22	3.85	4.93
	(2.01)	(1.11)	(1.73)	(1.33)	(1.82)	(1.49)	(1.38)	(1.20)	(2.24)	(1.39)
6	4.83	5.80	4.79	5.52	4.76	4.72	4.89	4.28	3.61	3.06
	(1.88)	(.76)	(1.17)	(1.19)	(1.74)	(1.80)	(1.80)	(2.01)	(1.87)	(1.34)
7									4.53	4.39
								(1.50)	(1.09)	

Table 37
Grade 2 Student's Triad Choices (Reasons)

Triad	1	2	3	4	5	6	7	8	9
Pre	3(ll)	3(ll)	3(ll)	3(ll)	3(vt)	1(vt)	2(vp)	1(vt)	2(ll)
Int	1(cl)	3(ll)	1(pr)	3(cl)	3(sz)	1(vp)	2(sz)	2(vtp)	3(vt)
Post	1(cl)	3(vp)	2(vtp)	3(vt)	3(vt)	1(vtp)	2(sz)	2(vtp)	3(vtp)

ll = looks like, Level 1
sz = size, Level 1
vp = visual components, Level 1
vt = visual transformation, Level 1
vtp = visual transformation of components, Level 1
pr = properties, Level 2
cl = classification, Level 3

Example of visual transformation:
Triad 5 (post). B and C. Because they are both the same thing except they are turned a different way.

Examples of visual transformation of components:
Triad 3 (post). A and C. They'd be the same if this line were a little bit straighter on the top.
Triad 8 (interim). A and C. Because A looks like C if these were just pushed in like that.
Triad 4 (pre). B and C. Because the square is exactly like a rectangle except it's not as spread out as a rectangle is.
Triad 4 (interim). B and C. These two, B and C. Because C is really a rectangle. I don't know why but my teacher just told us that.
Triad 4 (post). B and C. Because it's sorta like a square because if you went like that it would be a square.
Triad 6 (pre). A and C. No, I mean A and B. Because this one is sorta like this if you turn it like that way, the way this one is facing.
Triad 6 (interim). A and B. Because this one looks like it's just a little bit bigger and the bottom line is just slanted a little and the other little line [left side] here.
Triad 6 (post). A and B. Because this sorta looks like [it] if the line [left side] were straight, [then] the line on the bottom were straight.

Table 38
Number of Components of Motions Mentioned, as Indicators of Level

Student says:	# Components attended to:	Student is at least at Level:
Slides		
Move it to the right	1 out of 2	1
Slide it to the right	1 out of 2	1
Slide 3 spaces	1 out of 2	1
Move it to the right 3 spaces	2 out of 2	2
Slide it to the right 3 spaces	2 out of 2	2
Turns		
Turn it to the right	1 out of 3	1
Turn it 90 degrees	1 out of 3	1
Turn it halfway	1 out of 3	1
Turn it right 90 degrees	2 out of 3	1
Turn it right 90 degrees using this point as turn center	3 out of 3	2
Flips		
Turn it over (flip)	0 out of 1	1
Flip it over	0 out of 1	1
Flip it so sharp point goes from left to right side	0 out of 1	1
Flip over the vertical side	1 out of 1	2

Table 39
Level Determinations of Case-Study Students Before and After Motions Unit (From Lewellen, 1992, p. 131)

Student	Grade	Polygons Levels		Motions Levels	
		Interim	/ Final	Interim	/ Final
Kurt	6	1	1.5	1	1.5
John	6	1	1	1	1
Kelly	6	2	2	1	2
Allison	6	1.5	1.5	1	1.5
Jerome	4	1	1.5	1	1.5
David	4	1	1	1	1
Sandra	4	1	1	1	1
Helen	4	1	1	1	1

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